TWO PICTURES OF THE ITERATIVE HIERARCHY

by

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Abstract

I identify three reasons for holding the iterative conception to be the most beneficial conception of set. I then investigate two aspects of the iterative conception; the actualist and the potentialist picture. The potentialist picture has its origin in Aristotle, while the actualist picture stems from Cantor. Thus, both Aristotle's view about the potential infinite, and Cantor's theory of the transfinite and the Absolute is explicated. I evaluate the two pictures in relation to the reasons put forth in the beginning and argue that the two pictures both faces challenges that weakens the beneficial character of the iterative conception. In my investigation, I also identify what I claim is the core disagreement between the actualist and the potentialist, and hold this to show that the two pictures are not so different as one first gets the impression that they are. In the end I argue that when this core disagreement is identified, a reason is also identified, to prefer the one picture over the other.

Introduction

The intuitive understanding of what it means for something to be infinite, is that there is no end to it; it in some way goes on and on without ever stopping. This understanding has deep roots going back to the ancient philosophers and their characterizations of the infinite. Mathematics and the philosophy of mathematics, however, saw one of the greatest revolutions in time, with Georg Cantor, and his new measurements for measuring infinite collections; the transfinite numbers. When the infinite, thought to be without and end or limit, is shown to be measurable by a number, that means, it is shown to be *bounded*, how is its nature to be understood? Must the unmeasurable character of the infinite be rejected? Or is it our view on what things are infinite that must change? How are we really to understand the concept of *being unmeasurable* when mathematics ensures the existence of infinite collections?

Over the past century, philosophy has seen different ideas as to what the universe of mathematical objects is like. The idea of the infinite as something limitless is still embraced by the majority of the philosophical audience. However, there are several disagreements about how one are to answer the questions put forth above. This has led to apparently incompatible views concerning the nature of infinity and its unmeasurable structure. Two such views will be treated in this essay. But, some preliminaries will be needed before looking closer to it.

Set theory is the mathematical theory of *sets*. Sets are well-defined collections of objects called *members* or *elements*. A *pure set* is a set whose elements are all sets; throughout 'set' will mean 'pure set'. In set theory, sets and the properties of sets, are given axiomatically. The axioms of set theory implies the existence of a very rich mathematical universe, such that all mathematical objects can be construed as sets. For this reason, and because the set theoretic language allows for a formalization of all mathematical concepts and arguments, set theory is regarded as the foundation for mathematics. And since the theory of finite sets are equivalent to arithmetic, we can say that set theory essentially is the study of *infinite* sets.

Both of these aspects of set theory, as a foundation for mathematics and as the study of the infinite, are of great philosophical interest. In connection to the foundational aspect, the main philosophical debate has concerned the justification for the axioms accepted as the basic principles of mathematics, i.e. the axioms of Zermelo-Fraenkel set theory (ZFC). It is reasonable to question why exactly these axioms are the foundational principles of mathematics. Is it only for pragmatic reasons, because they accidentally imply a rich mathematical universe of sets successfully? Or are there other reasons to believe these principles to be the true principles of mathematics?

Cantor, regarded as the founder of set theory, showed with Cantor's Theorem, that holding the universe of sets itself to be a set, leads to contradiction. Together with Russell's discovery of the paradox within *naive set theory*, famously known as *Russell's paradox*, and the Burali Forti paradox, Cantor's discovery of the inconsistency of a universal set, constitute what is now known as the set-theoretic paradoxes. These paradoxes has caused a lot of worry for philosophers and mathematicians, and a main virtue with the axioms of ZFC is their apparent consistency.

A different question, related to the questions articulated above, concerns the difference between sets and *proper classes*. Since the universe of sets cannot itself be a set, then what is it? Cantor himself named it an *inconsistent multiplicity*. Today, pluralities of objects unable to form a set are known as proper classes, and some philosophers hold a proper class itself to be a collection. This however, has showed itself problematic, and to be analogues to the problems connected to a universal set. However, the philosophical debate has evolved around whether or not proper classes exist. If they do, what are they, and how are they to be implemented in the set theory? If they don't, how are we then to understand the universe of sets to be like?

In relation to both of the two issues, our understanding of what a set is has showed itself important. Two different conceptions of set has been elaborated, *the iterative conception* and the *limitation of size conception*. These conceptions of set are claimed to motivate, or justify, some, if not all, of the axioms of ZFC. This makes a conception of a set an essential aspect of set theory. If a conception of set successfully justifies the axioms of a set theory, the axioms are shown not to be just arbitrary principles of mathematics, but to have some sort of solid ground.

My focus here is on the iterative conception, which was properly introduced for philosophers by George Boolos in 1971. It has received broad recognition and is arguably regarded as the most plausible conception of set. Investigating this conception of set is interesting for several reasons. To see how it can work as a justification for set theory is one thing. A justification for the axioms of ZFC will after all give us a reason to believe that these axioms are actually *true*. But the iterative conception also tells us a story about sets in the set-theoretic universe. Reading this story and understanding what it actually tells us can give us comprehensive knowledge about what the set-theoretic universe is like. This is of great importance and interest, since essentially, a description of what a universe of sets is like, is a description of what *the infinite* really is.

The iterative conception describes a picture of the infinite, which by philosophers is interpreted in two different ways; I call these different interpretations *the actualist picture* and *the potentialist picture*. The potentialist picture originates from Aristotle, but saw its contemporary revival with Charles Parsons in the 1970's. The picture has developed and progressed in recent years, especially with the work of Øystein Linnebo. The actualist picture originates from Cantor and from the revolution in his name, that is said to have actualized the infinite.

These two pictures are arguably both tenable interpretations of the iterative conception. Still, they give two apparently incompatible characterizations of what the set-theoretic universe is like. My overall aim for this essay is to get to a proper understanding of what these two different aspects of the conception really amounts to; what are their differences? And how do these differences affects the benefits ascribed to the iterative conception?

To get there, I will, in chapter 1, explain what the iterative conception of set is. In the literature I find that people hold there to be three specific reasons for holding the iterative conception to be the most beneficial conception of set. I present these as *the three requirements* the iterative conception is said to fulfill. In chapter 2, I give a presentation of the actualist and the potentialist picture. This presentation is further elaborated in chapter 3, where the two picture's proper historical background is discussed. Thus, Aristotle's concept of the infinite and Cantor's theory of the transfinite and the Absolute is explicated. In chapter 4, I evaluate how the two different pictures attend to the three requirements put forth in chapter 1. I try to show that both pictures are tenable interpretations of the iterative conception, but that they both meet serious challenges. I conclude by showing how the understanding of what it means to be *unmeasurable* is the core disagreement between the two pictures, and how this both give us a reason to prefer the one picture over the other, and a reason to see the pictures as less different than was first assumed.

Chapter 1

The iterative conception of set

1.1 What it is

The iterative conception of a set is defined by Gödel as "something obtainable from the integers (or some other well-defined objects) by iterated application of the operation 'set of'".[1]

This can formally be read as

$$V_{\alpha+1} = \mathcal{P}(V_\alpha),$$

where the α is defined as a level in the process of set generation and is defined as the power set of all sets available at lower levels. Gödel notes that the 'iterated application' is meant to include transfinite iteration. That means that the totality of sets formed after finite iteration is itself a set and treated as available for the operation "set of"at the next level. Let λ be any limit ordinal reached, then $V_\lambda = \bigcup_{\gamma<\lambda}(V_\gamma)$

In an informal manner, the iterative conception holds a set to be any collection formed at some level or stage, where the stages are said to constitute a *set-hierarchy*. At the bottom, you find stage 0, with all individuals, or non-sets. At stage 1, all sets of individuals available at stage 0 is formed, and at stage 2, you have all individuals at stage 0, but also the sets formed from these at stage 1 available. Thus at stage 2, all sets formed from the available objects are formed. And so the process continues.

A great advantage with the iterative conception is that it gives a natural and intuitive explanation of what a set is and how sets are generated at the same time as it is consistent.[2]. Naïve set theory was shown to be inconsistent by

[1]Kurt Gödel, "What is Cantor's continuum problem?", (1964), in *Philosophy of mathematics. Selected readings*,(ed.by Benacerraf, P. and Putnam, H., Cambridge University Press, 1983), 474-475

[2]No inconsistency of the conception is known by now at least.

Russell's paradox, something which led to it passing away as a central option in the discussion of the foundations of mathematics. The theory also includes an axiom stating the existence of a *universal set* containing *all* sets, including itself, something which is a rather problematic statement.

The inconsistency of naïve set theory is traditionally seen to derive from the *principle of Naïve Comprehension*, which states that for any condition ϕ, there is a set whose members are exactly the things that satisfy ϕ.

The principle of Naïve Comprehension
$\exists y \forall x (x \in y \leftrightarrow \phi(x))$, where y does not occur free in ϕ

Russell's paradox is produced by letting ϕ be the condition of being all and only those sets that are not members of themselves. Consider then whether or not the set that satisfy ϕ is itself a self-membered set or not. In fact, if it is self-membered, then it is not. And if it is not self-membered, it is. This is a contradiction.

By its infinite hierarchical structure, the iterative conception avoids Russell's paradox. At each stage in the iterative set hierarchy, sets are formed from individuals and sets from lower stages. Thus, all sets will automatically satisfy the condition ϕ. They will all be sets that are not members of themselves, since a set cannot be formed before its elements. Also, since no set can be a member of itself, there can be no set containing *all* sets either, on the iterative conception. A set including all sets also includes itself, so a universal set would be self-membered, something which is not possible for sets in the iterative hierarchy.

A potentially problematic feature with the iterative conception is discussed by Gödel. He argues that there is a possibility for a problem of circulation for the iterative conception, regarding the theory of *ordinal numbers*.

> ...in order to state the axioms for a formal system, including all the types up to a given ordinal α, the notion of this ordinal α has to be presupposed as known because it will appear explicitly in the axioms. On the other hand, a satisfactory definition of transfinite ordinals can be obtained only in terms of the very system whose axioms are to be set up.[3]

The ordinals, were defined by Cantor as order types of well-ordered sets. The von Neumann definition of ordinals, which says that an ordinal is the set of its predecessor, is however, the ordinary understanding of ordinals today. The

[3]Kurt Gödel, "The present situation in the foundations of mathematics", in Gödel, K, *Collected works, Vol. III*, (Oxford University Press, 1995), 46

circularity claim can be addressed to the iterative conception on either understanding of the ordinal numbers.

The ordinal numbers stretches the iterative hierarchy from the finite to the infinite realm, and thus, the infinitude of the hierarchy is provided by the ordinals. Let an ordinal be defined as the order type of a well-ordering. For some things to be well-ordered, a collection of these things are usually required. A collection is the best way of representing a well-ordering, and the whole idea of well-orderings stems from thoughts about sets and collections. Thus, the theory of ordinals seems to require the concept of a collection for its definition. And that the concept of some kind of collection, or of a set, is required for the definition of the ordinals is obvious in the von Neumann definition, where the ordinals themselves are sets.

Thus, the iterative conception may be consistent, but it seems to be somewhat circular. However, Gödel himself offer a solution to the problems of circularity.

The idea from Gödel is to "build" the theory of ordinals inside the iterative hierarchy. Axioms for the two or three first stages of the set-generating process are not in need of ordinals to be defined, and these stages suffice to define very large ordinals. The idea is thus to:

> ...define a transfinite ordinal α in terms of these first few types and by means of it state the axioms for the system, including all classes of type less than α. (Call it S_α). To the system S_α you can apply the same process again, i.e., take an ordinal β greater than α which can be defined in terms of the system S_α and by means of it state the axioms for the system S_β including all types less than β, and so on.[4]

Instead of relying on an *external* theory of ordinals, one can define the ordinals within the system of set-generation itself and thus, avoid circularity for the iterative conception.

Sets and classes

The iterative set hierarchy, in its pure form, consists of sets, and nothing else. Thus, on the iterative conception, all collections are sets. Since there exist no universal set embracing *all* sets, the hierarchy is open-ended.

However, set theory make use of such collections as *the collection of all set* and *the collection of all ordinal numbers*, which on the iterative conception are said not to exist. As will be discussed in chapter 4, there seems to be good reason

[4]ibid

for not wanting to dispense with the use of *proper classes* is set theory. For this reason, there has been many attempts to make sense of them.

The concept of a proper class stems from a Cantor's notion of an *inconsistent multiplicity*. In a letter to Dedekind the 3rd of August, 1899, Cantor emphasize a distinction between consistent and inconsistent multiplicities.

> A multiplicity can be such that the assumption that all of its elements 'are together' leads to a contradiction, so that it is impossible to conceive of the multiplicity as a unity, as 'one finished thing'. Such multiplicities I call absolutely infinite or inconsistent multiplicities. As we can readily see, the 'totality of everything thinkable', for example, is such a multiplicity; later still other examples will turn up. If on the other hand the totality of elements of a multiplicity can be thought of without contradiction as 'being together', so that they can be gathered together into 'one thing', I call it a consistent multiplicity or a 'set'[5].

An inconsistent multiplicity is a plurality that it is impossible even to conceive of as being collected together. Consider Cantor's own example; everything thinkable. Let 'everything thinkable' be the plurality tt. For tt to form a set T, all the elements in tt must be "together", that means, they must coexist. Assume they do, and that tt forms the set T. T is then the set containing everything thinkable, which also must be something thinkable. But then T itself should be an element in the plurality tt, which is not the case, since T does not exist until tt is formed into a set. Thus, T is not the set of everything thinkable, and tt is not a coexistent plurality.

To make the distinction between consistent and inconsistent multiplicities more vivid, let a multiplicity be consistent only in the case where all its elements can be put into a box. Now, the problem with an inconsistent multiplicity such as everything thinkable is that all the elements of it cannot be put into a box. Every time the presumably last element is put into the box, a new element appears outside it. Putting this element inside the box won't help, since as soon as you do so, a new element again appears outside it.

As will be clear in chapter 3, there are different ways in which Cantor's notion of an inconsistent multiplicity is to be understood. The developed notion of a proper class has also seen different attempts of characterization. A prominent suggestion has been to suggest that classes, even though different from sets, still constitute a unity. Thus, on this view, classes are understood as a new kind

[5]Georg Cantor, Letter to Dedekind, 3rd of August, 1899, in W.B. Ewald, *From Kant to Hilbert, Volume 2: A Source Book in the Foundations of Mathematics*, (Oxford University Press, 2005), 931-932

of set-like entities. The objection to this, is that it only displaces the original problem of universality. For what about the class of all classes? This question takes us right back to the original position.

Thus, it is a challenge for philosophy to make sense of proper classes, and on the iterative conception, it is either held to not exist at all, or exist only in a potential sense. This will be further elaborated in the next chapter.

1.2 Why the iterative conception?

In set theory, sets and their basic properties are given axiomatically. With different axiomatic set theories follows different ideas of what sets are. It seems reasonable that one's view of what sets are, and how they behave, is a view corresponding to the conception of set expressed by the axiomatic set theory one holds to be true. However, if this is so, one may wonder what a conception of set, such as the iterative one, really is to deliver. If there, in any case, is the axiomatic theory that puts restrictions on what we believe about the nature and properties of sets, it is not so easy to see what importance the conception really has.

However, both philosophers, logicians and set theorists have claimed that there are several benefits of importance related to the iterative conception. On of them is that the conception actually gives us some evidence for many of the axioms of ZFC. The set theorist and philosopher D.A. Martin, for instance, notes that:

> The iterative concept suggests the axioms of Zermelo-Fraenkel set theory. Indeed, those axioms should be thought of as an attempt to axiomatize the iterative concept rather than an attempt to approximate the inconsistent concept.[6]

A similar claim is found in Boolos (1971) and (1989), where Georg Boolos claims that the iterative conception *motivates* or *justifies* most of the axioms of ZFC. Boolos shows how this motivation is fulfilled by deriving the axioms of Z[7], except that of extensionality, from his *stage theory*, which is a theory supposed to precisely express his "rough description" of the iterative conception.[8]

[6]Donald A. Martin,"Set Theory and Its Logic by Willard van Orman Quine", in *The Journal of Philosophy*, Vol.67, No.4, (1970), 112

[7]Z is Zermelo set theory, a sub theory of both ZF and ZFC. ZF is Z plus the axioms of replacement, and ZFC is ZF plus the axiom of choice.

[8]Boolos' "rough description" is similar to the informal characterization of the iterative conception given above.

Boolos treats the terms 'motivation' and 'justification' synonymously, and throughout, if not anything else is specified, I will use the term 'motivation' in this way. It is not entirely clear what is meant by the claim that the conception *motivates* or *justifies* the set theory. However, I find it reasonable to suppose that to motivate some of the axioms of ZFC involves providing some evidence for these axioms. This means that the iterative conception is held to deliver an important epistemological benefit; it gives us a reason to think that some of the axioms of ZFC are true.[9] Also, if the conception can provide an evidence for some of the axioms, it can in principle do so for further set-theoretic axioms as well.

A second advantageous aspect of the iterative conception is expressed by Boolos.

> ZF alone (together with its extensions and subsystems) is not only a consistent (apparently) but also an independently motivated theory of sets: there is, so to speak, a "thought behind it" about the nature of sets which might have been put forth even if, impossibly, naive set theory had been consistent.[10]

Such A "though behind" the axioms of ZFC shows that the axioms are not arbitrary principles. They are not developed for purely pragmatic reasons, that is, because "they work" in being a foundation of mathematics. Such a pragmatic motivation can be found for other, and different axiomatic systems as well. Rather, by pinpointing a "thought behind" the set theory, the iterative conception provides an informative and *uniform* characterization of what set theory is about, which both provides a unification and systematization of the axioms.

As Boolos also notes, the iterative conception is an idea about sets that could have been put forth even if naive set theory had been consistent. For this reason, it is regarded an *independent motivation*. That is of course, independent of the wish to avoid the set-theoretical paradoxes. Thus, the characterization of the hierarchy of sets that the iterative conception provides is independent of the wish to avoid paradox. One can say that it is a characterization developed from some kind of Rawlsian *Original Position*, behind a *Veil of Ignorance*. On this position, there is no knowledge of the set-theoretical paradoxes, such that the naive conception may as well be considered consistent. Boolos' point is that even behind such a Veil, one would still think the characterization the iterative conception provides to be true.

[9]By 'us' is not meant everyone, but rather set theorists or semi-professionals within the field of set theory and philosophy of mathematics

[10]Boolos, "The Iterative Conception of Set" in *Logic, logic and logic*, (Harvard University Press, 1999), 17

A third and last benefit of the iterative conception is, as has been earlier noted, that it provides a response to the set-theoretic paradoxes.

Now, what aspects of the iterative conception accounts for these different benefits? There must be certain qualities of the conception that makes it able to both function as a motivation for axioms of ZFC, as a uniform characterization of what set theory is about *and* provide a satisfactory response to the paradoxes of set theory. I identify three reasons that I find are being held responsible for the benefits of the iterative conception.

An actual conception

Martin gives a reason as to how the iterative conception can be said to motivate the axioms of ZFC.

> ...when one does set theory (other than the semantics of set theory) one normally thinks in terms of the intuitive concept and not the formal axioms".[11]

Thus, Martin claims that the iterative conception is a conception *in use* by mathematicians, or at least that it corresponds to a type of set-theoretical thinking, that is in use. This means, it is an *actual* conception. Set-theoretical thinking, according to Martin, is done in terms of the conception, and not in terms of the axioms. That means, the iterative conception corresponds to the level of set-theoretical thinking where developments within the field is actually done.

This is also the idea one gets from Gödel.

> This concept of set, however, according to which a set is something obtainable from the integers (or some other well-defined objects) by iterated application of the operation "set of", not something obtained by dividing the totality of all existing things into two categories, has never led to any antinomy whatsoever; that is, the perfectly "naïve" and uncritical working with this concept of set has so far proved completely self-consistent.[12]

According to Gödel, the iterative conception is a naive conception of set that is consistent and mathematically respectable. It corresponds to a set-theoretical way of thinking that has been in use for a long period of time, and has so far proved itself consistent.

Also Boolos seems to hold the actuality of the iterative conception to be a reason to show the conception to be a plausible one. In his defense of why

[11] Martin, "Set Theory and Its Logic by Willard van Orman Quine", 113

[12] Kurt Gödel, "What is Cantor's continuum problem?", 474-475

the axiom of extensionality should not be seen as guaranteed by the iterative conception, he notes that:

> ...our aim, however, is to analyze the conception we have , and not to formulate some imperfectly motivated conception that manages to imply the axioms.[13]

Both Martin, Gödel and Boolos then, seem to agree, that the iterative conception is an *actual* conception of set, or corresponds to a set-theoretical way of thinking that is being practiced my mathematicians, is a reason why the conception is able to motivate (most of) the axioms of ZFC. Thus, we can identity this as a requirement the iterative conception is said to fulfill, and which contributes to the conception's status as an advantageous conception of set.

***(1)*:** The iterative conception is an actual conception, or corresponds to a set-theoretical way of thinking that is being practiced.

A natural/intuitive conception

Boolos emphasize the *natural* character of the iterative conception, which he explains thus:

> "Natural" here is not a term of aesthetic appraisal [...] but simply means that, without prior knowledge or experience of sets, we can and do readily acquire the conception, easily understand it when it is explained to us, and find it plausible or at least conceivably true.[14]

Thus, a conception's naturalness is essentially linked to ease of acquisition of the concept and plausibility. I find Boolos' use of the notion 'natural' to be approximately similar to the use of the term 'intuitive' in this same context.[15] If one informally defines what it is for something to be intuitive, as something like: *without previous reflection on the something in question, one can immediately understand and apprehend it*, the two notions at least seem to be intimately connected.

Even though the characterizations of a *natural* or *intuitive* conception is rather vague, I find that there is a more or less clear idea behind the idea in

[13] George Boolos, "Iteration again" in *Logic, logic and logic*, (Harvard University Press, 1999), 93

[14] Ibid, 89

[15] Martin uses the two terms synonymously

question. Recall what Gödel says in the passage above, that the iterative conception is a *naive* conception. This characterization points to the fact that the conception is one untutored by careful mathematical thinking. It is not a result of complex mathematical work, but rather something that is more or less easy to grasp and easy to work with. A useful comparison is the role the number line plays in understanding the Peano axioms. Isolated, the axioms are informative, but some mathematical maturity is required to understand only from the axioms, what the arithmetic properties of the natural numbers are. However, considering the number line, it is obvious that it has an intuitive character, that even makes children understand the properties of the natural numbers.

The natural character of the iterative conception, claims Boolos, distinguish it from other conceptions, such as the *limitation of size* conception. A general character of the limitation of size conception is that it is built upon the principle: *Some things form a set if and only if there are not too many of them*. However, there are different versions of the conception, based on different understandings of what it means to be "to many". Boolos' point is that limitation of size-principle makes the conception an *unnatural* one, since "one would come to entertain it only after one's preconceptions had been sophisticated by knowledge of the set-theoretic antinomies".[16]

Thus, the claim from Boolos is that the natural character of the iterative conception accounts for its independence from the wish to avoid the set-theoretic paradoxes. It seems reasonable that one, behind a hypothetical Veil of Ignorance concerning set-theoretical reasoning, could have put forth the iterative conception as a plausible conception of set, simply because it is so easily understood and apprehended.

Boolos' claim is supported by Martin, in his discussion of Quine's axiomatic set theory New Foundations (NF).

> New Foundations is not the axiomatization of an intuitive concept. It is the result of a purely formal trick intended to block the paradoxes. No further axioms are suggested by this trick. Since there is no intuitive concept, one is forced to think in terms of the formal axioms. Consequently, there has been little success in developing New Foundations as a theory.[17]

New Foundation is obtained from taking the axioms of the Type Theory, and erasing the type annotations. One of the axioms is the comprehension schema, but stated using the concept of a stratified formula, and makes no reference to types. Quine acknowledges himself, in Quine (1995) that his theory is

[16] Boolos, "Iteration Again", 90

[17] Martin, "Set Theory and Its Logic by Willard van Orman Quine", 113

solely motivated by the wish to avoid paradox. The lack of a different motivation than the desire for consistency, has resulted in NF's lack of development, claims Martin. The theory lacks an intuitive concept, that means, a concept easily grasped and acquired. Thus, one is forced to think in terms of the axioms.

Thus, the intuitive character of the iterative conception account for the independent status of the motivation it gives to axioms of ZFC. However, that the conception has this natural character, which makes it easy to grasp and apprehend, and which, according to Martin, set theorists thinks in terms of when doing set theory, then this natural aspect of it also accounts for the uniform "thought behind" the axioms the conception is said to provide.

A second requirement the iterative conception is said to fulfill then, can be formulated:

(2) *The iterative conception is an intuitive, or natural conception.*

An explanatory value

As Gödel notes in the passage above, the "perfectly "naïve" and uncritical working with this concept of set has so far proved completely self-consistent".[18] As was shown in the previous section, the iterative conception does provide a response to the set-theoretical paradoxes. The hierarchical structure of sets on the iterative conception provides a reason for why sets such as the Russell set and the universal set do not occur. Since a set is formed only from elements formed at earlier stages than the stage where it's at, no set can contain itself, and thus, none of the paradoxical sets occur.

According to Boolos, that the iterative conception gives an *explanation* as to *why* the paradoxical sets do not occur accounts for the response to the paradoxes.

> A final and satisfying resolution to the set-theoretical paradoxes cannot be embodied in a theory that blocks their derivation by artificial technical restrictions on the set of axioms that are imposed *only because* paradox would otherwise ensue; these other theories survive only through such artificial devices.[19]

Again, it is Quine's theory New Foundations that is under attack.[20] Since a theory like NF is developed for the purpose of avoiding the set-theoretic paradoxes, it gives no explanation as to why the paradoxical sets do not occur. It

[18] Gödel, "What is Cantor's continuum problem?", 475
[19] Boolos, "The Iterative Conception of Set", 17
[20] And also his ML

just ensures that there are no paradoxes, but provides no response to why this is the case.

Thus, the response to the paradoxes given by the iterative conception has an *explanatory value* that is of importance. This gives us the third requirement that the conception is said to fulfill.

(3) *The iterative conception provides an explanatory response to the set-theoretical paradoxes*

Now, these are three different requirements which each contribute to the different benefits of the iterative conception. That the conception is actual accounts for the motivation it is held to give to axioms of ZFC. That it is an intuitive conception accounts for the independent aspect of this motivation. However, it is reasonable to suppose that the conception could maintain its status as a motivation, even if it was not a natural conception (this is the case for the limitation of size conception). However, the natural character of the iterative conception also enables the conception to pinpoint a "thought behind" the axioms, and thus contribute to a kind of unification of the principles of the set theory. Lastly, as we saw above, the explanatory value that is ascribed to the conception enables the conception to give a satisfactory response to the set-theoretical paradoxes.

Chapter 2

Actualism and potentialism on the iterative conception

Before explicating the two different readings of the iterative conception, a preliminary reflection may be useful. Recall that the naive conception of set, with its *Principle of naive comprehension* (NCO) were showed to be inconsistent by the discovery of Russell's paradox. Again, the principle states that for any condition ϕ, there is a set whose members are exactly the things that satisfy ϕ. Now, the naive conception of a set is a logical conception of a set, and NCO is thus usually treated as a principle about concepts or properties, and sets. However, Stephen Yablo has argued that, if pluralities are given a place in between properties and sets, NCO can be seen as the product of two principles, which without problem can be put into the iterative context.[1] The two principles are

Naive Plurality Comprehension (NPC)
For any property P, there are the things that are P

Naïve Set Comprehension (NSC)
Whenever there are some things, there is a set of those things.

Now, if one, as the naive conception does, hold both principles to be true, Russell's paradox is easily generated. However, logicians and philosophers has responded to Russell's paradox in two different ways. One response has been to reject NSC, holding NPC to be true. The other response rejects NPC, while holding NSC to be true. In what follows we will see that these two different ways of responding to the paradox are reflected in the two different ways of understanding the iterative conception.

[1]Stephen Yablo, "Circularity and Paradox" in *Self-reference*, (2004)

2.1 The actualist picture

The *actualist picture* of the iterative set hierarchy is favored by many philosophers.[2] The actualist claims that all the sets in the hierarchy exist actually. This idea is better explained by looking at how the actualist understands the language used in the explication of the iterative conception. As is evident from the informal characterization of the conception in the previous chapter, when explicating what the conception is, we make extensive use of a language of time and activity. Sets are *formed* at *stages*. At each stage, all sets formed at stages *earlier than* the stage you're at are *available* for set formation.

A central question connected to understanding the iterative conception is how this generative vocabulary is to be understood. A literally understanding more or less implies a constructivist reading of the set hierarchy, which is not to be desired.[3]

The actualist Boolos claims the language of time and activity used on the iterative conception is to be understood as a mere metaphor, and that it is "thoroughly unnecessary"[4] for explaining the iterative conception. Boolos points to an observation made by Dan Leary, about how the metaphor may arise from a narrative convention. That means , when explaining the conception, one naturally starts with mentioning the individuals or the null set, then the set that contains the individuals or the null set and so further on, since this is how the sets are arranged. It would be unnatural to start from a different and arbitrary stage in the hierarchy, even though it is possible to do so.

> The fact that it takes time to give such a sketch [of the set hierarchy], and that certain sets will be mentioned before others, might easily enough be (mis-)taken for a quasi-temporal feature of sets themselves.[5]

Thus, the narrative convention that motivates the temporal metaphor gives an explanation of why one may take the formation talk literally, and ascribe a temporal feature to set-existence. It is, according to Boolos , possible to explain the iterative conception by just replacing the terms 'stage', 'is formed at' and 'is earlier than' with 'ordinal', 'has rank' and 'is less than', or simply by a listing of the sets in the hierarchy.

[2]George Boolos, (1971) and (1989) and Gabriel Uzquiano (2003) are the main sources for my account of the picture

[3]See Parsons (1977) for a discussion on this. I'm assuming a platonist framework for my discussion of the iterative conception

[4]Boolos, "Iteration again", 91

[5]Ibid, 90

> ...there are the null set and the set containing just the null set, sets of all those, sets of all *those*, sets of all *Those*,...There are also sets of all *THOSE*. Let us now refer to these sets as "those." Then there are sets of those, sets of *those*,...Notice that the dots "..." of ellipsis, like "etc.," are a demonstrative; both mean: *and so forth*, i.e., in *this* manner forth.[6]

The idea is clear, the sets are already *there*, and they are existent. To talk about sets continuously coming into existence does not make sense on this account. All the sets formed on the iterative conception are, on the actualist account, *already* formed.

In relation to what were said in the previous chapter, it is clear that the set hierarchy on the actualist account cannot be described as a Cantorian inconsistent multiplicity. The inconsistent multiplicity is inconsistent for the reason that the elements it consists of do not coexist. They cannot all together be put into the same box. Since all the sets in the actualized hierarchy coexist, they can, however, be put into the same box. But, while the inconsistent multiplicity fails to form a set exactly for the reason that all its elements cannot come together in the same box, the actualized hierarchy, even though considered consistent, still cannot form a set. This means, that on the actualist picture, the iterative hierarchy exist as an actualized plurality.

It may seem puzzling that some things can exist "as many", but not "as one". However, that the actualist holds this to be the case for the set hierarchy implies that, on the actualist account, the distinction between that of being a plurality and that of being a set is a substantial ontological one. Take for instance an existing tree. If there is a substantial ontological difference between a plurality and a set, then the fact that the tree exists is not a sufficient condition for it to be a member of its singleton. For that to happen, it is required that the tree's singleton exists *in addition* to the tree itself, and that it bears a certain relation to the tree.

That there is such an ontological difference is explicitly stated by Boolos. Considering some Cheerios in a bowl, he notes:

> ...is there, in addition to the Cheerios, also a set of them all? And what about the $> 10^{60}$ subsets of that set? (And don't forget the sets of sets of Cheerios in the bowl.) It is haywire to think that when you have some Cheerios, you are eating a *set*–what you're doing is: eating THE CHEERIOS.[7]

[6] Ibid, 91

[7] George Boolos, "To Be is to Be the Value of a Variable", in *Logic, logic and logic*, (Harvard University Press, 1999), 72

Thus, for a plurality to form a set, something more than just the existence of the plurality is required. This explains why the set hierarchy can be said to exist as an actualized plurality, but not form a set.

Also, Boolos, in Boolos (1984), argues that plural quantification is ontologically innocent and that this makes the reading of sentences such as

(S) There are some sets which are all and only the non-self-membered sets,

which is often formalized

(S') $\exists R \forall x (Rx \leftrightarrow x \in x)$

not committed to a *class* of all sets. ,

It has been debated how one, on a formalization such as S', is to understand the second-order quantifier $\exists R$. A suggestion is that it ranges over a class of all sets. However, on the actualist view, this is undesirable, since on this account, *all sets* exist, but not the class of them. Boolos also shows that plural quantification is interdefinable with monadic second order logic and thus suggests to interpret the quantifier $\exists R$ as a plural quantifier, such that it is better written $\exists rr$. On the assumption that plural logic is ontologically innocent, this makes it possible to quantify over *all sets* without committing oneself to the existence of any class of all sets.[8]

A last characteristic fact of the actualist picture is connected to the preliminary remark made above. On what has been said about the actualized structure of the iterative hierarchy, we see that, on the actualist account, there exist more pluralities than sets. Thus, the actualist rejects the principle of *naive set comprehension* (NSC). Even though holding that there is a substantial ontological gap between the existence of a plurality and the existence of a set explains the intelligibility of the actualist characterization of the set hierarchy, a reasonable question to the actualist is: *why* do some pluralities not form a set? To give an answer to this question has proved difficult and we will see in chapter 4 that it proves a serious challenge to the actualist picture.

The actualist picture rejects NSC, but it embraces the principle of *naive plurality comprehension* (NPC). This is also evident from the fact that the hierarchy is held to be an actualized plurality. Because the elements of the hierarchy are coexistent, there is a definite fact of the matter what sets there are in the hierarchy.

[8] It is, however, arguable whether plural quantification is ontologically innocent.

2.2 The potentialist picture

The *potentialist picture* of the iterative set hierarchy holds, contrary to the actualist, that the sets in the hierarchy is not coexistent, and for this reason, that the hierarchy exist potentially. As we will see in the next chapter, the picture stems from Aristotle's idea of the potential infinite, but it was first introduced in a set-theoretical context by Charles Parsons. However, in Parsons (1977), Parsons uses Cantor and Cantor's notion of an inconsistent multiplicity in explaining his view of the iterative hierarchy. About Cantor's definition of an inconsistent multiplicity, he notes:

> It is noteworthy that Cantor here identifies the possibility of all the elements of a multiplicity *being together* with the possibility of their being collected together into one thing. This intimates the more recent conception that a 'multiplicity' that does not constitute a set is *merely potential*, according to which one can distinguish potential from actual being in some way so that it is impossible that all the elements of an inconsistent multiplicity should be actual[9]

The idea from Parsons is that Cantor identifies the possibility of being a plurality with the possibility of being a collection, or a set. When doing so, he says, it is reasonable to suggest that an inconsistent multiplicity exists potentially; it cannot be formed into a set, and neither can the elements it consists of coexist.

We saw that the actualist holds it to be a substantial ontological difference between that of being a plurality and being a set. Parsons interpretation of Cantor suggests that the potentialist holds a view to the contrary; that there is no such substantial ontological difference between the existence of a plurality and a set. This is emphasized by Parsons:

> A multiplicity of objects that exist together *can* constitute a set, but it is not necessary that they *do*. Given the elements of the set, it is not necessary that the set exists together with them. [...] However, the converse does hold and is expressed by the principle that the existence of a set implies that of all its elements.[10]

Thus, as long as the objects of a plurality coexist, their existence imply the *possibility* of a set of these objects.

In chapter 1 it was noted that the notion of an inconsistent multiplicity can be described as a plurality with elements that can't be put into a box together,

[9] "Charles Parsons, "What is the iterative conception of a set", in *Philosophy of mathematics, Selected readings*, (1983), 514

[10] 526

since at least one element always will remain outside the box. Parsons presents a more moderate interpretation of Cantor's notion, holding that an inconsistent multiplicity is such that whatever elements of the multiplicity are in the box, the *possibility* of there being an element outside it, is always there. However, it is not *necessary* that it is.

From Parsons interpretation of inconsistent multiplicities, we get the idea that on the potentialist account, there is not a definite matter, what are the elements of the multiplicity. For instance, what the potential picture says about the iterative hierarchy is that it is not a definite matter what are *all sets*. In connection to the preliminary remark made i the beginning of the chapter, we see that the potentialist picture rejects the *principle of naive plurality comprehension* (NPC). This means that, on the potentialist account, there are cases where a property P fails to define a plurality, even though P is a determinate property. This may seem rather controversial. If P is a determinate property, it means that it is a determinate matter for any x, whether x has P or not. However, Stephen Yablo, in Yablo (2004) calls attention to the rejection of NPC as a response to Russell's paradox, and he notes that:

> How then can it fail to be a determinate matter what are *all* the things that has P? I see only one answer to this. Determinacy of the P's follows from
>
> i determinacy of P in connection with particular candidates
>
> ii determinacy of the pool of candidates
>
> If the difficulty is not with (i), it must be with (ii).[11]

And like Yablo, the potentialist holds that whenever a determinate property P fails to define a plurality, there is an indeterminacy of "the pool of candidates".

Øystein Linnebo claims in Linnebo (2010) that this indeterminacy is explained by the potential character of the iterative hierarchy. Since a set is an immediate possibility given the existence of its elements, there is no way the elements of the hierarchy can exist "together", and thus no definite matter what actually are the sets of the hierarchy.

Even though this potential interpretation may be a plausible reading of the iterative conception, it is still a fact that the language used on the conception use expressions such as sets being *given* or *available* for set formation, which, as noted above, has undesirable consequences if taken literally. Parsons, however

[11] Yablo, "Circularity and paradox", 152

suggests that we "replace the language of time and activity by the more bloodless language of potentiality and actuality"[12]

Parsons suggestion has recently been elaborated, by Linnebo, in Linnebo (2010) and (2013a). Linnebo holds that there are, on the iterative conception, two different ways of understanding the set theoretic quantifiers. A non-modal way and an implicit modal way. One should typically analyze the use of set theoretic quantifiers in the implicit modal way.

> So when a set theorist says that a formula holds for "all sets," she should typically be understood as claiming that no matter how far the hierarchy of sets is continued, the formula will hold of all the sets formed by then. And when a set theorist says that a formula holds for "some set," she should typically be understood as claiming that it is possible to continue the hierarchy of sets such that there is some set of which the formula holds.[13]

This implicit modal character can be made explicit through a translation from the language of non-modal set theory to the language of modal set theory with modal quantifiers. We introduce the two modal operators $\Diamond$ and $\Box$. $\Diamond\phi$ is usefully interpreted as "it is possible to go on to form sets so as to make it the case that ϕ" and $\Box\phi$ as "no matter what sets we go on to form it will remain the case that ϕ.[14]

Inconsistent multiplicities as Fregean concepts

The potentialist picture may give a consistent and plausible account of what a set hierarchy existing potentially is like. However, we are still to question how we best are to understand what a potential entity is. Parsons suggestion is thus:

> I want to suggest that predication plays a constitutive role in the explanation of Cantor's notion of multiplicity as well and that at least an "inconsistent multiplicity" must resemble a Fregean concept in not being straightforwardly an object.[15]

To understand the suggestion from Parsons it is important to understand Frege's ontological distinction between functions and objects. The distinction

[12]Parsons, "What is the iterative conception of set?", 526

[13]Øystein Linnebo, "Pluralities and Sets", in *Journal of philosophy*, Vol.107, No.3, (2010), 155

[14]Linnebo emphasizes that the modal notions involved must not be understood as ordinary metaphysical modality. Since the modality invoked allows the existence of sets to be contingent, it must be a more fine-grained metaphysical modality.

[15]Parsons, "What is the iterative conception of set?", 516

is exhaustive, thus, something is either an object or a function, and nothing is both. Physical things and truth values are examples of objects, while concepts and mathematical functions are examples of functions.

One can think about a function as some sort of machine with several wholes for inputs on the one side, and a single whole for outputs on the other side. You feed the machine with arguments through the input-wholes, and it spits out a value through the output-whole. Multiplication $(\cdot)$ is one sort of mathematical function. It takes some numbers, say 4 and 5, as its arguments and give out one number, $(4 \cdot 5 = 20)$ 20, as its value.

The distinction between functions and objects is, however, mirrored in the language, by the distinction between incomplete and complete names or expressions. A predicate is an example of an incomplete name and a sentence an example of a complete name. This distinction can be analyzed in the same manner as the ontological one. Predicates, like functions, have argument places, and when fed with arguments they give out a value, more specifically a truth value.

Further, the different ontological types and the types in the language match each other. The complete names name objects; sentences, for instance, name truth values. Incomplete names name functions, and Frege holds a concept to be the reference of a predicate. As mentioned above, a concept is a function. More specifically, claims Frege, it is a function "whose value is always a truth-value".[16]

Thus, a predicate names a concept, and an inconsistent multiplicity is seen by Parsons to resemble a concept. Parsons explicates his suggestion further:

> In the Cantorian context, predication seems to be essential in explaining how a multiplicity can be given to us not as a unity, that is as a set. Much the clearest case of this is understanding a predicate. Understanding 'x is an ordinal' is a kind of consciousness or knowledge of ordinals that does not so far 'take them as one' in such a way that they constitute an object.[17]

Thus, understanding the predicate "x is an ordinal" involves a knowledge of the ordinal numbers that does not assume the existence of *all* ordinals. The concept of *being an ordinal* is not dependent on the fact that all things that are ordinal numbers coexist. In other words, we can have the concept and understand it without having all the instances of it.

It is reasonable to hold a set to be an object, in the Fregean sense. Thus, as Parsons point out, one argument that favors the identification of an inconsistent multiplicity with a concept, is that, since we know the multiplicity is not a set, it

[16]Gottlob Frege, "On concept and object", P. T. Geach, and Max Black, *Mind*, (1951), 139

[17]Parsons, "What is the iterative conception of set?", 516

is not an object. Since the ontological distinction is exhaustive, the inconsistent multiplicity is a concept.[18].

However, Parsons holds there to be other reasons to claim the distinction between set and inconsistent multiplicities to be an ontological one.

> From the idea that a set is constituted by its elements, it is reasonable to conclude that it is *essential* to a set to have just the elements that it has and that the *existence* of a set requires that of each of its elements.[19]

The idea that a set is constituted by its elements is the cornerstone of the iterative conception. Consider again the box-example. It is essential for a set x that *all* the elements it consists of can be put inside a box. If there is an element a, which is a member of x, but is left outside the box, then the elements inside the box do not constitute the set x.

Parsons point is that an inconsistent multiplicity is essentially different than a set in this sense: The inconsistent multiplicity exists, even though all its elements will never be put in a box. There will always be a possibility of one element outside the box, and thus, it is not essential to the inconsistent multiplicity to have "just the elements that it has". It can exist without there being a definite matter what elements it has.

For this reason, claim Parsons, one should hold hat inconsistent multiplicities are intensions, and Fregean concepts, and that they should not be construed realistically, like sets are construed.

We can compare Parsons understanding of a concept with the nominalist. The nominalist holds that predicates do not designate anything, but that the understanding of a predicate is anyway independent of whether or not it has a designatum. Parsons, however, holds intensions to exist, but not in the same sense as objects exists. While an object has a life independent of its definition, an intension has a linguistic correlative and is dependent on this correlative to exist.

Thus, on the potentialist picture, we are to understand the iterative hierarchy as something that are potential in its nature. In contrast to a set, the hierarchy does not consist of a fixed range of objects and thus, should not be represented by an extensional entity. The better way of representing the hierarchy, is by way of a Fregean concept, which is of an intensional nature.

[18]Ibid, 517

[19]Ibid, 519

Chapter 3

From the ancient to the contemporary concept of infinity

The potentialist claims that the set hierarchy exists as a potential entity because he holds it to be an indefinite matter what are the sets it consists of. The actualist, on the contrary, holds the hierarchy to be something determinate, and claims it exists as an actualized plurality. In this chapter, we will see that these two different views on the hierarchy each has its very important historical origin, which are arguable a necessary source for getting an appropriate understanding of the different pictures.

3.1 Aristotle on infinity

An important question for Aristotle when investigating the infinite, is whether there exists any infinite body. That means, whether any single thing or magnitude can be infinite. Aristotle's answer to this is negative, and he provides several arguments to show that the idea of a body both infinite by addition and by division is not possible. The arguments given for ruling out the possibility for a body being infinitely big is mainly empirical, while the arguments for rejecting the existence of a body infinite by division appeals to the paradoxes of the infinitely small, such as Zeno's paradox of Achilles and the tortoise, and the paradox of the divided stick.

However, even though Aristotle concludes that a magnitude is not infinite, neither by division nor addition, he still claims there are infinite magnitudes.

> ...to suppose that the infinite does not exist in any way leads obviously to many impossible consequences: there will be a beginning

> and an end of time, a magnitude will not be divisible into magnitudes, number will not be infinite.[1]

So a magnitude must nevertheless be infinite by division. Thus, it may seem like Aristotle claims that the infinite both exists and does not exist, something which is a plain contradiction. However, Aristotle's solution to this seemingly contradictory considerations is that the infinite exists *potentially*. When he concludes that the infinite cannot exist, it is the *actual* infinite that is under consideration.

To understand how the implementing of a potential infinite solves the problems of infinity for Aristotle, it is useful to first look at his fundamental distinction between potentiality and actuality.

The distinction is a metaphysical one. Consider a lump of bronze, and a bronze statue made of that exact lump. In the case of the lump one can say that the bronze is *potentially* a statue, and that it is *actually* a statue only when it has the form of the statue. Thus, the lump exists potentially as a statue because it has the quality of possibly becoming a statue. It is natural to view the potential existence as an *incomplete* existence, since really, what it means to exist potentially is to exist as something not yet actualized or completed. This does not mean that the lump of bronze is an incomplete lump of bronze. Its existence as the lump of bronze is the*actual* existence, while its potential existence as statue is only an additional, but not actualized existence.

Also, the potential existence as a statue is not unique for the lump of bronze. It potentially exists as a cymbal, a ship propeller or as a bunch of jeweleries as well. In contrast, it does not potentially exist as a cake, flower or a volleyball net. Thus, there are certain features of an entity such that in virtue of these features, the entity is potentially something else.

This metaphysical distinction between potentiality and actuality in Aristotle is, however, not fully applicable to his distinction between what is potentially and actually infinite. The bronze is potentially a statue because it can *become* or *be* a statue. Thus, what *is* potentially is so because it can *be* actually what it is potentially. This is not the case with what's infinite. "[S]omething infinite will not be in actuality", claims Aristotle. What's infinite is therefore bound to always be potentially.[2] This means that being potentially infinite is structurally different from being potentially a statue.

[1] Aristotle, *Physics*, book III, ch.6, in *The complete works of Arsitotle, The revised Oxford translation*, Vol.1, ed. by Jonathan Barnes, (Princeton University Press, 1984), 351

[2] This has been debated. See Jaakko Hintikka,"Aristotelian infinity", in *The Philosophical Review* (Vol75, No.2, 1966) and Jonathan Lear, "Aristotelian infinity" in *Proceedings of the Aristotelian Society*, (Vol. 80, 1979)

As we saw, the lump of bronze exists potentially as a statue *because* it has the property of possibly existing as a statue. The potential infinite lacks this property; the infinite's existence as the infinite is potential and incomplete, but this is not because it, in virtue of some features that it has, can become the infinite in the actualized and completed sense. Rather, following Aristotle's reasoning, we see that the potential existence of the infinite is constant and can never be actualized, an that the actual infinite is an impossibility.

The potential infinite

Consider the paradox of Achilles and the tortoise, which appears when considering a footrace between Achilles and the tortoise. Achilles runs twice as fast as the tortoise and lets the tortoise start a certain distance ahead of him in the race. Before Achilles can overtake the tortoise, he must reach the point where the tortoise started. When he does that, the tortoise has advanced half the distance that initially separated them. Now, Achilles must make up this distance, but by the time he does that, the tortoise will again have advanced some distance. Thus, whenever Achilles reaches a point where the tortoise has been, he still has farther to go. And since there are infinite number of points Achilles must reach, where the tortoise has already been, Achilles can never overtake her.

This is paradoxical, since, considering the information one has about the speed they hold, it will be possible to calculate precisely how long it will take Achilles to actually overtake the tortoise from the start of the race. But, if one holds that Achilles can overtake the tortoise, one is committed to hold that he has, when running the distance, actually passed over infinitely many points.

A different paradox, which also alludes to the incoherent idea of something physical being divided into infinitely many parts, is the paradox of the divided stick. Imagine that one cut an infinitely divisible stick in half at some time, and that half a minute later, each of the two halfs again are cut in half. Then, after another half a minute, each quarter of the stick is cut in half, and this process continues *ad infinitum*. Now, what would be left at the end of the minute? infinitely many pieces? and would each piece have any length?

Aristotle claimed that the source for these paradoxical situations, was that, in our reasoning about them, one assumes the infinite both to be actual *and* infinite. Something infinite is *actual* if all of it actually exists. That means, all its parts or elements have to coexist; as Aristotle notes, for something to be actually, it must be "a definite quantity"[3]. And it is this definiteness, that makes the actual infinite a victim of the paradoxes of the infinitely small.

[3]Aristotle, *Physics*, book III, ch.6, 348

Now, the potential infinite is free of this definiteness. Aristotle characterize it thus:

> ...generally the infinite has this mode of existence: one thing is always being taken after another, and each thing that is taken is always finite, but always different.[4]

Something infinite must be understood as an entity of some sort, divisible into parts. If you take one part, or one thing, there is always another to be taken. Thus, if you are to go through all the parts of the entity, the process of going through them will, in principle, never stop.

This characterization is reflected in the ancient concept of infinity, *apeiron*, which means 'without an end' or 'without a limit'. According to Aristotle, there are two ways this term is being used. One way is its application to what is not capable of being gone through, because, for whatever is infinite "it is not its nature to be gone through". Secondly it is applied to what "naturally admits of a traversal but does not have a traversal or limit."[5] Thus something infinite, according to Aristotle, is by nature impossible both to be gone through and to be traversed.

This untraversable aspect of the infinite is emphasized by Aristotle.

> .,..something is infinite if, taking it quantity by quantity, we can always take something outside. On the other hand, what has nothing outside it is complete and whole. [...] Nothing is complete which has no end and the end is a limit.[6]

It is useful to understand what is being said by Aristotle, in terms of boxes, as was done in connection to Cantor's consistent and inconsistent multiplicities. In box-terms, one can read Aristotle as saying that when something infinite is taken quantity by quantity, that is, when part by part of the infinity is put into a box, some part will always remain outside the box. The process of putting quantities into the box has no end to it, and thus will never be completed.

As was noted above, the metaphysical concept of potentiality found in Aristotle can be understood as that of *being* in an incomplete sense. A lump of bronze exists potentially as a statue because it can become a statue, but does not actually exist as a statue. The lump's existence as a statue is thus incomplete. However, as was made clear, the metaphysical distinction between potentiality and actuality does not fully parallel that between the potential and actual infinite. Following Aristotle's descriptions of the infinite, in contrast to the lump

[4] ibid, 351

[5] ibid, 347

[6] ibid, 352

of gold, something is potentially infinite precisely because it can never become actually what it is potentially.

The infinite is thus incomplete in a more fundamental way than what the lump of bronze is. The incompleteness of the lump of bronze existing as a statue is possibly temporary, since the lump of bronze may be formed into a statue. If it is, its incomplete existence as a statue becomes complete. The infinite, on the other hand, can never move beyond the state of incompleteness. It is incomplete because there is no possibility for it to be completed.

An interesting point from Jonathan Lear illuminates how we are to understand the idea of the potential infinite.

> ...it is easy to be misled into thinking that, for Aristotle, a length is said to be potentially infinite because there could be a process of division that continued without end. Then it is natural to be confused as to why such a process would not also show the line to be actually infinite by division. However, it would be more accurate to say that, for Aristotle, it is *because* the length is potentially infinite that there could be such a process.[7]

The point from Lear is that it is not the unending process of divisions applied to a magnitude that accounts for the magnitude being potentially infinite. If it was, there is no good reason as to why the unending process in question is not actually infinite, since holding that it exists to account for the potentially infinitude of the magnitude implies that it is *there*. And if the unending process in fact is *there*, there is no reason as to why one cannot explain the infinite by just listing all the elements it consists of, and the avoid the use of any language of time and activity.

However, as Lear notes, it would be more correct to claim that for Aristotle, it is the potential infinitude of the magnitude that accounts for the unending process. But neither this is entirely correct, since the point in question is that a process of division that continue without end *does not exist*. For a process of division continuing without end can exist, it must have the ability to be carried out. However, if someone is to carry it out, the process will at some time end, since this someone must be mortal. And at the time it ends, there will still remain division that could have been made. Thus, such a process cannot be carried out and it cannot exist.

From this Lear concludes that a "length is potentially infinite not because of the existence of any process, but because of the structure of the magnitude"[8].

[7] Lear, *Aristotelian Infinity*, 193

[8] ibid

And the structure of the magnitude, is such that for any division of it, possible divisions remain "unactualized".

Now, this idea of the potential infinite avoids the paradoxes of the infinitely small. When holding the infinite to exist potentially, one is not committed to the fact that Achilles has passed over infinitely many points, if he overtakes the tortoise. One can hold instead, that the distance Achilles ran, when overtaking the tortoise, is not *actually* infinite. It is rather the case that however many divisions that is recognized on this distance, it will always be possible to recognize more. Thus, the distance Achilles runs is not divisible into actually infinitely many parts, but the process of dividing it is infinite in the potential sense. That means, it has no end or limit to it, and the divisions can in principle go on infinitely.

It is the same for the divisible magnitude. It is not the fact that one can, by a process of division, separate a magnitude into infinitely many parts, since this would amount to the magnitude having an actual infinity of parts. The case is rather such that the process of dividing the magnitude in principle will go on without stop. Regardless of how many divisions that are made, it is always possible to make another one. In other words, there is no end to the process of division, and thus, the magnitude has a potential infinity of parts.

The temporal metaphor

The potential infinite is characterized as something which is such that "one thing is *always* being taken after another", and something that *has no end* to it. These are temporally loaded characterizations of the infinite. We have seen that on the modern understanding of the infinite, there is a great wish to dispense with what is characterized as a metaphoric language of time and activity. The actualist, with Boolos, claims the metaphoric language is "thoroughly unnecessary", while the potentialist claims it must be replaced by a modal language. Would it be possible for Aristotle to dispense with this temporal language without losing anything essential from the description of the infinite?

A.W. Moore argues that to dispense with the temporal metaphor on Aristotle's account of the infinite is impossible, since the appeal to time is not seen as a metaphor, but rather taken literally by Aristotle.[9] For Aristotle, questions of possibility and impossibility are intimately connected to questions of time. Thus, asking whether or not something is possible, is akin to asking whether or not it at some time would be the case.

Thus, neither the more moderate rejection of the metaphoric language, that of the modern potentialist, can be applicable to Aristotle's explication of the in-

[9]A.W. Moore, *The Infinite*, 2nd edition, (Routledge, 2001), 40

finite, since what Parsons characterize as the "bloodless language of potentiality and actuality" is for Aristotle intimately connected to time. Since possibility for Aristotle is possibility *in time* a replacement of the language of time and activity with that of potentiality and actuality is not a substantial replacement.

3.2 Three important developments

The idea of the actual infinite as something problematic which leads to paradox, was a prevailing understanding for centuries, and did not really change until the ideas of Cantor. Before going through some of the main points of Cantor's theory of the infinite, it is useful to call attention to three essential developments, from the ancient to the contemporary understanding of the infinite, that came with the Cantorian revolution.

The first development is connected to the *principle from Euclid*. The principle holds that a whole is always greater than its parts, and where a principle treated as an obvious truth by the ancient philosophers. Recall how Aristotle characterizes the infinite's mode of existence as such that one thing is always taken after another, "and each thing that is taken is always finite, but always different." If a part of the infinity where infinite, and not finite, this would not make sense, since "to be infinite and the infinite are the same [...] But the same thing cannot be many infinities".[10] Thus, because two infinite entities are of the same size, according to Euclid's principle, an infinite entity cannot be part of another infinite entity.

Now, Cantor formally proved that a great many infinite sets where the same size. The comparison of size rested on a one-to-one correspondence between sets. A surprising result of his formal work was a proof showing that some infinite sets, being proper subsets of another set, were the same size as their superset. That is for instance the case for the set of all even natural numbers and the set of the natural numbers. The set of even natural numbers is contained in the set of natural numbers. Cantor formally proved that even though this is so, the two sets are still of the same size, something which is a rejection of Euclid's principle.

The second development, or difference, between the ancient and contemporary idea, is seen as the real upshot from Cantor's formal treatment of the infinite, namely the proofs made to show that not all infinite sets are the same size. Cantor's Theorem says that for any set S, the set of all subsets of S, that is, the power set $\mathcal{P}(S)$ of S is greater in size than S. That means, first of all, that there are infinite sets bigger than others.Secondly, it means that sets of im-

[10] Aristotle, *Physics*, Book III, ch.6, 348

mensely great size exists.

Lastly, the third development from the Aristotelian to the Cantorian understanding of the infinite, is the meaning of the term 'infinite'. As we saw, Aristotle took the meaning of the term literally, and on the his understanding, to be infinite, meant to be *without an end* or *without a limit*. This was synonymous with being *larger than any natural number*. After the Cantorian revolution, however, the meaning of the term changed. In mathematical practice and in philosophy today, to be infinite is still understood as being larger than any natural number. However, as has been noted, Cantor formally proved that infinite sets were of different sizes; thus, larger than natural numbers, but still possible to measure by other numbers, namely the *transfinite numbers*. These numbers will be further explicated in the next section. However, what this means is that there are many infinite sets that are larger than any natural numbers, but still can be measured, and thus has a limit, or a bound. In this sense, the term 'infinite', on the contemporary understanding, do not only apply to things that cannot be measured or bounded.

These three developments from the ancient to the contemporary theory of the infinite led to an *actualization* of the infinite. Cantor's transfinite is seen as an *actually* realm of the infinite, and transfinite sets are bounded by the transfinite numbers. Thus, the idea of the infinite existing potentially close to vanished with the Cantorian revolution. However, as we saw, Parsons uses Cantor as a main source of inspiration for his potentialist view. We will see in the sections below that there is a tension in Cantor's account of the infinite, that can favour both the actualist and potentialist.

3.3 Cantor's theory of the infinite

The ordinal numbers

Of great importance in Cantor's theory of the infinite are the ordinal numbers. In *Grundlagen* Cantor introduces the concept of a well-ordered set. A well ordering of a set A is a linear ordering under which every non-empty subset of A has a least element. To each well ordered set there corresponds an order type, which is the way in which the set is ordered. If two well ordered sets $(A, <_A)$ and $(B, <_B)$ have the same order type, there is a one-to-one correspondence c of the members of A with the members of B. This means that $x <_A y$ if and only if $c(x) <_B c(y)$.

Two different orderings of the same set can, for instance, have the same order type. Consider the set of all natural numbers. The order type of $\mathbb{N}$, ordered by $<$ is ω. Now, if all odd numbers are placed after all even numbers, and the order is otherwise unchanged, this well ordered set

$$0, 2, 4, 6, 8, ...1, 3, 5, 7, 9, ...$$

has the order type $\omega + \omega$. The same set, but ordered such that the multiples of four comes after the rest of the natural numbers

$$1, 2, 3, 5, 6, ...0, 4, 8, 12, 16, ...$$

will have the same order type, $\omega + \omega$.

An *ordinal number* is the order type of a well ordered set, and given any well ordering, there is an ordinal specifying its order type. The ordinals fulfill this function of measuring well ordered sets via a well-ordering of themselves. In *Grundlagen*, Cantor used what is known as *the genrating principles* (principle (ii) and (iii) below) as a definition of the ordinal numbers. He later forsook these principles as a *definition* of the ordinals, because the principles were not purely mathematical. He then defined the ordinals as order types of well ordered sets, and this is the definition of ordinals used today.[11] However, the generation principles are of great philosophical interest, especially in connection to whether or not one should interpret Cantor as a potentialist or an actualist, which will be discussed below.

We can find three substantial principles for what ordinals exist in the early work of Cantor.

(i) The first ordinal is the order type of the empty set. We call this '0'

(ii) If α is an ordinal, there is a new ordinal number $\alpha + 1$ which is the immediate successor of α

(iii) Given any definite sequence of consecutive ordinals there is a first ordinal, a 'limit', after the sequence. No ordinal number smaller than this limit can be strictly greater than all ordinals in the given sequence.

The conditions (ii) and (iii), the *the generating principles*, specify the two ways ordinals are being generated. Thus, ordinals are generated by proceeding to the successor $\alpha + 1$ from an ordinal α, and by proceeding to the first ordinal after an endless succession. We call the first ordinal after an endless succession a limit-ordinal. The first limit-ordinal is ω.

The generating principles extend the natural number sequence by the ordinals and stretches the sequence into the infinite. It follows from the two principles that there is no definite sequence containing *all* the numbers. To see this,

[11] See Ignasi Jané, "Idealist and Realist Elements in Cantor's Approach to Set Theory", in *Philosophia Mathematica*, (2010) for thorough discussion on this.

let a definite sequence of ordinals be given. If it contains *all* the numbers, then it must have a largest element, α. But by the first generating principle (ii), α has an immediate successor, $\alpha + 1$, which is not in the original sequence. Thus, the sequence cannot contain all numbers.

If, on the other hand, we are given the sequence of all numbers, but it has no largest element, the second generating principle (iii) tells us that it then exists a number strictly larger than all the numbers in the sequence. This ordinal, however, is not itself in the sequence either, so the sequence cannot contain all numbers.

Since there exists no definite number sequence containing all ordinal numbers, it is reasonable to suppose the sequence of all ordinal numbers is an *indefinite* sequence. But what is meant with 'definite' here? As Ignasi Jané points out, that in *Grundlagen*, 'definite' is consistently used in contrast to 'variable'. The contrast Cantor draws between the potential and actual infinite is that between the *variable* finite, which is the *improper* infinite, and the actual, *definite* and *proper* infinite. A *definite sequence* is thus either a finite or an actually infinite sequence, while an *indefinite sequence* is a sequence neither finite nor actually infinite.

The fact that the two generating principles have no closure - that no actually existing totality can contain all ordinal numbers, fits neatly in with Cantor's later distinction between consistent and inconsistent multiplicities. One of the multiplicities Cantor considered as inconsistent was precisely *all ordinal numbers*. Considering this multiplicity, which is also characterized as an *extended number sequence*, as an outcome of the two generating principles, yields a natural understanding of its inconsistency. It follows from the principles themselves that the elements of the multiplicity cannot coexist. Thus, it is not only by definition that a multiplicity such as all ordinals are inconsistent, it also follows from considerations about the existence of the elements in the multiplicity itself.

The transfinite and the Absolute

Cantor makes a fundamental distinction between what he called *the transfinite* and *the Absolute*. The transfinite and the Absolute constitute the infinite, but are different infinities. Cantor called every infinite set a transfinite set. What distinguish the transfinite and the Absolute is, amongst other things, the fact that the transfinite is increasable, while the Absolute is not.

Cantor says little about what it means for something to be increasable, but there is an intuitive way in which we understand some transfinite domains to be increasable. Consider the domain formed by the natural numbers. We know that this domain is contained in the domain formed by the rational numbers, which again form part of the domain pointed to by the real numbers. Thus, in a numerical way, one can characterize the natural number-domain as increasable to the rational number-domain, which again is increasable to the real number-domain.

Another way to look at it, is to consider the fact that the transfinite is subject to numerical determination. We saw that the ordinal numbers extended the natural number sequence and gave way to a numbering also of well ordered infinite sets: the transfinite ones. The smallest transfinite ordinal number is ω, which can be increased to the next greater transfinite ordinal number, which is $\omega + 1$, which can be increased to to the next largest ordinal, $\omega + 2$, and so on. Also, the *cardinal numbers*, which measure the size of finite and infinite sets and constitute another extended number sequence, enumerate transfinite sets.

Thus, the transfinite is marked out by either or both of the two number sequences. The Absolute, which is unincreasable, is not represented in any of the number sequences and is rather symbolized by the whole of either of them. As Michael Hallett notes, if 'increasable' means numerical increasability, and numerable means being representable in the transfinite number sequence, then "the Absolute must be both 'beyond' mathematical numeration *and* unincreasable."[12] Just consider how the whole ordinal sequence, or the whole cardinal sequence, contains everything denumerable. Thus, the sequences cannot themselves have a number. The sequences mark the order or size accretion. Where there is no number to mark neither order nor size, there cannot be any accretion. Thus, the Absolute is beyond both number and increasability.

3.4 A tension between actualism and potentialism

We saw that Parsons referred to Cantor and his distinction between consistent and inconsistent multiplicities when motivating a potentialist reading of the iterative set hierarchy. Recall how Cantor himself defines an inconsistent multiplicity.

> For a multiplicity can be such that the assumption that *all* of its elements 'are together' leads to a contradiction, so that it is impossible to conceive of the multiplicity as a unity, as 'one finished thing'. Such multiplicities I call *absolute infinite* or *inconsistent multiplicities*.[13]

[12] Michael Hallett, *Cantorian set theory and limitation of size*,(Oxford: Clarendon Press, 1984), 43

[13] Cantor, Letter to Dedekind, in W.B. Ewald, *From Kant to Hilbert, Volume 2: A Source Book in the Foundations of Mathematics*, (Oxford University Press, 2005), 931-932

It is a fact that the elements of an inconsistent multiplicity are not all existent together. If each element individually, however, can be said to exist, then a natural understanding of the multiplicity is that there are elements of it that in some way *can* exist, just not together with all of the others. And this is the intuitive feeling we get when we consider multiplicities such as "everything thinkable". As soon as you have gathered all the elements of the multiplicity in one box, another element that should have been inside the box reveals itself outside it. A coexistence of all the elements of the multiplicity is impossible and a potential explication of the situation seems accurate.

However, the notion of an inconsistent multiplicity is part of what creates a tension between actualism and potentialism in Cantor's works. The distinction between consistent and inconsistent multiplicities, on the one side, together with the idea of the generating principles, support a potential reading of Cantor, while what is called *the domain principle*, on the other side, is a clear actualist principle, and is a strong argument in favour of an actualist reading of Cantor.

The domain principle

The *domain principle*[14] is formulated by Cantor as follows:

> Every potential infinite, if it is to be applicable in a rigorous mathematical way, presupposes an actual infinite.[15]

The potential infinite is, as noted earlier, characterized by Cantor as the *variable finite* and *improper infinite*, and as an opposite to the *determinate*, *definite* and *proper* infinite.[16]

So what does it mean that the improper infinite in all cases of existence presupposes the proper infinite? The explanation from Cantor is this:

> There is no doubt that we cannot do without *variable* quantities in the sense of the potential infinite; and from this can be demonstrated the necessity of the actual-infinite. In order for there to be a variable quantity in some mathematical study, the 'domain' of its variability must strictly speaking be known beforehand through a

[14]Hallett, *Cantorian set theory and limitation of size*,7

[15]Cantor, "Mitteilungen zur Lehre vom Transfiniten", parts I and II, in*Zeitschrift fur Philosophie und philosophische Kritik*, (1887),410-411, transl. in Jané, *Idealist and realist elements in Cantor's approach to set theory*, 22

[16]*Georg Cantor, "Foundations of a general theory of manifolds: a mathematico-philosophical investigations into the theory of the infinite*, (1883), in Ewald, *From Kant to Hilbert*, 882

definition. However, this domain cannot itself be something variable, since otherwise each fixed support for the study would collapse. Thus, this 'domain' is a definite, actually infinite set of values.[17]

Thus, the domain principle seems like a complete rejection of the potential infinite. Even though the potential infinite, as a variable finite must exist and is important for mathematical investigation, to every instance of it, there corresponds with necessity a domain not itself variable, since otherwise, the investigation would lack a solid basis. And since the domain in question must be both infinite and definite, this domain must be the actual infinite, more specifically, the *transfinite*.

Both Hallett (1984) and Jané (2010) have pointed to the fact that the domain principle is not only a principle of existence but also of intelligibility. The point is illustrated by Cantor when he says that "the potential infinite [...] always points to an underlying *transfinitum*, without which it can neither be nor be thought"[18]. This is to say that even for one to be able to grasp the idea of the infinite, the existence of the actual infinite is necessary.

Surely, the correspondence of an actual infinite to any potential infinite reduces the ontological status of the potential infinite. The most essential in Aristotle's analysis of the infinite is the idea of the infinite being potential and *not* actual. When the existence of any variable quantity, such as a potential infinite, with necessity requires a solid and actual basis, that means, when the potential infinite cannot exist without a corresponding actual infinite domain, it seems like the potential infinite is being reduced to the actual infinite and eliminated.

However, the potential infinite under investigation here, is the *variable finite*. It is reasonable to suppose the potential infinite reduced is the potential infinite at work in notions like endless addition of 1, the continuity in the real line etc, and not the overall infinite - the Absolute, containing also the whole transfinite. The Absolute is not characterized as an *improper infinite*, but rather, as what's "beyond both number and increasability". That the unincreasable Absolute presupposes the increasable transfinite does not seem right, something which is arguably justified by what may seem like an extended version of the domain principle:

> The transfinite, with its wealth of arrangements and forms points with necessity to an absolute, to the 'true infinite', whose magni-

[17] Georg Cantor "Über die verschiedenen Ansichten in Bezug auf die actualunendlichen Zahlen", (1886), transl. in Hallett (1984), 25

[18] Cantor to Schmid, 22 April 1887, quoted in Jané, *Idealist and realist elements in Cantor's approach to set theory*, 23

tude is not subject to any increase or reduction, and for this reason it must be quantitatively conceived as an *absolute* maximum.[19]

Just like the original domain principle required a corresponding transfinite domain for the existence of any potential infinite, Cantor here points to the necessity of a corresponding Absolute for the existence of any infinite sets. The Absolute, or the "true infinite", is the unincreasable stable basis needed for the existence of the increasable transfinite.

Jané claims that Cantor in the passage above presents the idea of the Absolute as an "all-encompassing universe". That means, the Absolute is characterized as itself a definite collection.[20] Hallett is more moderate in his interpretation, and notes that there is no clear evidence that Cantor conceived of the Absolute as itself a collection, and that the closest he comes saying it, is in the passage above.[21] Both of them discuss how this view of the Absolute fits in with Cantor's view of the Absolute as equivalent to God, and both argue that the view of a definite Absolute may have strong connections to Cantor's theological convictions. This, however, does not diminish the importance of the idea itself.

It seems obvious that Cantor claims that the transfinite points to something bigger and supposedly more stable than itself, as a presupposition for its existence. The Absolute, as Cantor argues, symbolizes the whole of the two extended number sequences, and thus, also symbolizes the *whole* of the transfinite. In this connection, it is natural to view the Absolute as some sort of universe for both the finite and the transfinite, that means, as a universe for everything mathematizable. However, that the Absolute is a definite universe contradicts the view of the extended number sequences as indefinite.

If it is right, as Jané and Hallett claims, that the passage from Cantor above says that the transfinite relies on a definite domain of *all* the transfinites for its existence, then there is a clear tension between an actualist and potentialist reading of Cantor. On the one side, the idea of inconsistent multiplicities, backed up by the two generating principles for ordinal numbers, amounts to the idea of multiplicities, such as everything transfinite, all ordinal numbers and all cardinal numbers, as multiplicities with not coexistent elements. These multiplicities are easily pictured as having a potential nature.

On the other side, the domain principle offers a characterization of the Absolute as a completed domain, necessarily existent for the existence of the transfinite. And such a characterization of the Absolute supports and actualist read-

[19]Cantor, "Mitteilungen zur Lehre vom Transfiniten", (1887), 405, translated in Jané, *Idealist and realist elements in Cantor's approach to set theory*, 24

[20]Janeé, *Idealist and realist elements in Cantor's approach to set theory*, 22

[21]Hallett, *Cantorian set theory and limitation of size*, 44

ing of Cantor, and contradicts the idea of inconsistent multiplicities. Since, a multiplicity is inconsistent for the reason that conceiving of it as "one finished thing" is impossible, since the elements of the multiplicity do not coexist. But if it is the case that the inconsistent multiplicities are all aspects of an actual, completed universe, then there is no reason why their elements should not coexist. It would then rather be natural to conclude that all multiplicities inhabits the same actual universe, and, thus, that all of them are consistent. That would mean rejecting the existence of any inconsistent multiplicities.

The adequate reading of Cantor?

In connection to the tension between actualism and potentialism in Cantor, to what extent should it affect a potentialist reading of his work, such as the one Parsons does in "What is the iterative conception of set?"?

Following Jané, the distinction between consistent and inconsistent multiplicities is tenable only if the definiteness of the Absolute is rejected. That means, one will have to let an arguably important part of Cantor's theory go, if the potential reading is to remain consistent. Whether or not rejecting the definiteness of the Absolute is tantamount to rejecting the actual domain principle is, however, uncertain. If it is, then this means rejecting an important aspect of Cantor's *realism* and to dismiss an element of the theory of the infinite, of which Cantor himself was very clear: Any potential infinite presupposes an actual one. This would be an undesirable result.

If, on the other hand, rejecting the definiteness of the Absolute means only rejecting the extended version of the domain principle, that claims there is a corresponding definite Absolute to the transfinite, the tension would be resolved, while the view of a corresponding transfinite to any potential infinite would still be intact. This would be a more reasonable rejection, since it is the definiteness of the Absolute that in fact creates the tension, and not the definiteness of the transfinite.

It follows from these considerations, that the idea of a definite Absolute universe is only tenable if the distinction between consistent and inconsistent multiplicities is rejected. Then one must hold that only consistent multiplicities exist. However, this also means rejecting an explicitly stated idea of Cantor's, and arguably also means rejecting the intuitive conclusion one draws from the fact that the generating principles have no closure: that the extended number sequence is indefinite.

In the end, then, the question of whether the potentialist or the actualist reading of Cantor is the adequate reading, is a question of which one of the ideas found in Cantor it would be best to reject. Rejecting the definiteness of the Absolute allows for a potential reading, while rejecting the distinction between consistent and inconsistent multiplicities allows for an actualist one.

Chapter 4

Two tenable pictures?

From what has been said in the two foregoing chapters it is clear that the actualist and potentialist pictures, even though they are both pictures or aspects of the iterative conception, give two very different characterizations of what the structure of the iterative set hierarchy is like.

To summarize the main points: The actualist holds the hierarchy itself to be a plurality of sets. Pluralities are extensional entities, so the hierarchy is viewed by the actualist as something extensional. On the potentialist's account of the iterative conception, the elements of the hierarchy are seen as incapable of existing together. For this reason, the plurality of all sets does not exist, and the hierarchy is, by the potentialist, rather seen to exist as an intensional entity.

We saw that this difference between the two pictures is also expressed in their different understandings of the generative vocabulary used on the iterative conception. Such an informal characterization supports the picture of an open-ended hierarchy.

> ...every set is formed at some stage of the following "process": [...] at stage 0 only the null set is formed. The sets formed at stage 1 are all possible collections of sets formed at stage 0, [...] Immediately after all stages 0,1,2,..., there is a stage, stage ω. the sets formed at stage ω are, similarly, all possible collections of collections of sets formed at stages earlier than ω. [...] There is no last stage: each stage is immediately followed by another.[1]

That there is no last stage and that each stage is immediately followed by another means that the process of set formation in principle never stops. However, the contemporary actualist and potentialist both want to do away with this generative vocabulary, but differ in how to understand it. The actualist holds

[1]Boolos, "Iteration again", 88

that the vocabulary used is "thoroughly unnecessary"[2] and is to be understood as metaphorical. The potentialist, on the other side, suggests, in the manner of Parsons, to replace it by "the more bloodless language of potentiality and actuality".[3]

In effect then, the hierarchy on the potentialist account is described as something ontologically different than on the actualist account. This is a substantial difference between the two, and it is unclear how two so radically different views can both be aspects of one and the same conception of set.

In what follows I will evaluate the two different interpretations of the iterative conception and I think the best way to do so is with a view to what the conception is claimed to deliver. As we saw in chapter 1, the iterative conception was claimed both to be an independent motivation for most of the axioms of ZF, to function as a uniform "thought behind" the axioms, and also to provide a response to the set-theoretic paradoxes. It was also held that these benefits of the conception, are due to three different qualities that it has. These were put forth as *the three requirements* the iterative conception is said to fulfill.

(1): *The iterative conception is an actual conception, or corresponds to a set-theoretical way of thinking that is being practiced.*

(2) *The iterative conception is an intuitive, or natural conception.*

(3) *The iterative conception provides an explanatory response to the set-theoretical paradoxes*

The following evaluation of the actualist and potentialist picture will discuss if and how the different pictures upholds the satisfaction of the three requirements. I believe this evaluation will show that both pictures are subject to great challenges, but most importantly, that it will also reveal that the two picture's very different characterizations of what the hierarchy is like, is the result of a disagreement about how the unmeasurable character of the infinite is to be understood. When this disagreement is located, it will be evident that the difference between the actualist and potentialist may not be as substantial as it first appears.

4.1 (1) An actual conception

There is one prominent worry related to requirement (1) above, holding that the iterative conception is an actual conception. This is the worry connected to

[2] Ibid

[3] Parsons, "What is the iterative conception of set?", 526

how the vocabulary of classes seems to play an important role in set-theoretic work today. Recall that there exist no classes on the iterative conception. However, there seems to be strong arguments in favor of keeping the vocabulary of classes in set theory, holding that it is impossible to dispense with. Gabriel Uzquiano, in Uzquiano (2003) points to how the use of classes in set theory provides what may be "irreplaceable formulations" of large-cardinal hypothesis.

I will look briefly at the use of the vocabulary of classes in formulating large-cardinal hypothesis.[4] I don't, however, have the specialist knowledge to give a qualified argumentative discussion of the matter in question, which requires mathematical proficiencies on a high level. However, I will try to present the idea of why some philosophers and set theorists claim that classes are difficult to dispense with in set theory.

As noted in the previous chapter, a *cardinal* or a *cardinal number* is said to measure the size of a set. It is however, also defined as an ordinal α that cannot be put into one-one correspondence with any smaller ordinal. A central area of mathematics has been to investigate what happens when one suppose that there are really large cardinal numbers. Zermelo introduced in Zermelo (1930) the smallest of the large cardinal numbers, what are called *inaccessible cardinals*.

> **Inaccessible cardinals:** Suppose there is a cardinal κ such that if we take less than κ many sets, which each is less than κ, then the union of these sets will be less than κ. Suppose also that whenever $\alpha < \kappa$, then $|\mathcal{P}(\alpha)| < \kappa$. That means, κ is larger than the power set of α. Then we say that κ is an *inaccessible* cardinal, and we can show that $V_\kappa \models ZFC$.

Later, the existence of still larger cardinals have been investigated and formulated, and a prominent way of doing so involves the concepts of an *inner model* and that of an *elementary embedding*.

Definition 1 *Let On represent all the ordinals. A class M is said to be an inner model just in case $M \models ZFC$ and $On \subseteq M$*

Definition 2 *Let V be the collection of all sets, M an inner model, and j be some one-one function from V to M which is not the identity function. Then we say that j is an elementary embedding from V to M, which is symbolized $j : V \to M$, just in case for all formulas in the language of set theory ϕ:*

$$V \models \phi(x_0, ...x_n) \leftrightarrow M \models (j(x_0), ..., j(x_n))$$

[4] I'm grateful to Sam Roberts, who is finishing his Ph.D. in philosophy of set theory at the University of Birkbeck, for his guidelines on this subject

In other words, an elementary embedding is an injective function from V to M that preserves all of first-order logic. Because j is not the identity function, we call it a *non-trivial* embedding.

Now, let the *critical point* of an elementary embedding j be the least cardinal κ moved by j, (i.e. for which, $j(\kappa) \neq \kappa$). After a certain point, almost all large cardinal hypothesis are either defined in terms of critical points of elementary embeddings or can be formulated in terms of the critical points. We call κ a *measurable cardinal* if it is the critical point of an embedding $j : V \to M$.

Large cardinal hypotheses formulated in terms of elementary embeddings are connected to each other in a specific way. By imposing additional closure conditions on the inner model M, the hypothesis that there is an elementary embedding $j : V \to M$ is strengthened. For instance, the hypothesis that there is a *measurable* cardinal is substantially weaker than the hypothesis that there is a *superstrong* cardinal, which is the critical point of an embedding $j : V \to M$ where $V_{j(k)} \subseteq M$.

Uzquiano points to how Kenneth Kunen (1971) exemplified a use of embedding principles in set theory when he proved the following theorem:

Theorem 1 *Assuming the axiom of choice, there is no* $j : V \to V$

The theorem is a central and important result in the investigation of large cardinals and it is taken to be highly *non-trivial*.

Now, if the iterative conception is to correspond to a set-theoretical way of thinking that is being practiced my mathematicians, which is the theory of ZFC, then the conception ideally should give some account of set-theorist's use of the proper classes j, V and M, but also pays respect to the non-triviality of theorems like Theorem 1. But how can this be done, when there exist no proper classes on the iterative conception?

One idea is to find equivalent set-theoretic formulations of large cardinal hypothesis, within first-order ZFC, that don't make use of the vocabulary of proper classes. A lot of set-theoretical work has been devoted to this, but Uzquiano presents three arguments holding that it will be unsatisfactory to use them as replacement for class-talk.

First of all, the fact that there are such equivalent set-theoretic formulations that do not make use of a vocabulary of classes may be a "source of comfort", holds Uzquiano, however, he notes that "the interest of the relevant large-cardinal principles stems more often from their model-theoretic characterization than from their technical formulations within first-order ZFC".[5]

[5]Gabriel Uzquiano, "Plural Quantification and Classes", in *philosophia Mathematica*, Vol.11, No.3, (2003), 71

Secondly, he notes that there is no *a priori* guarantee that all arbitrary large cardinal hypotheses stated with the help of a non-trivial elementary embedding admit of equivalent set-theoretical formulations within first-order ZFC. There is, for instance, little reason to think there exist equivalent formulations of the reversed hypotheses, such as the one holding the existence of non-trivial elementary embeddings, $j : M \to V$, from an inner Model M into V. To this point, Sam Roberts, a Ph.D. student in philosophy of set theory at the university of Birkbeck, London, has pointed out to me that interpreting class talk as first-order formulations does not respect the non-triviality of theorems like Theorem 1. This is also expressed by Joel Hamkins, who says this about the suggestion of interpreting class talk in first-order ZFC:

> Our view is that this way of understanding the Kunen inconsistency [our Theorem 1] does not convey the full power of the theorem. Part of our reason for this view is that if one is concerned only with such definable embeddings j in the Kunen inconsistency, then in fact there is a far easier proof of the result, simpler than any of the traditional proofs of it and making no appeal to any infinite combinatorics or indeed even to the axiom of choice.[6]

Thus, other than just having no a priori guarantee that all large-cardinal principles have equivalent first-order formulations, one has a reason to believe they don't.

Lastly, Uzquiano points to how large-cardinal hypotheses are often developed from a specific extension of the language of ZFC, which may be helpful, but is "not nearly as perspicuous"[7] as it would be, developing the study of the large-cardinal hypothesis with the use of proper classes. It is not obvious how this last argument is different from the first argument presented above, about how the interest of the large-cardinal principles most often stems from their model-theoretic characterizations. However, in light of Uzquiano's further comment, that "'set theorists often begin to work within an informal theory of sets and classes, and then search for technical formulations within either ZFC or some schematic extension thereof",[8] I take this last point to express that an informal set theory of sets and classes is more "intuitive" and easier to work with than the formulations of first-order ZFC or a schematic extension of this.

Thus, it seems like a more or less robust mathematical understanding, that the use of proper classes in set theory should not be dispensed with. I leave the

[6]Joel David Hamkins in Hamkins, Joel David, Greg Kirmayer, and Norman Lewis Perlmutter, "Generalizations of the Kunen inconsistency", in emphAnnals of Pure and Applied Logic, (2012), 1873

[7]Uzquiano, "Plural Quantification and Classes", 72

[8]ibid

discussion here by assuming that to do away with the vocabulary of classes in set theory is a problematic matter.

Can the iterative conception adjust to the set theory?

If it is not possible to do away with the vocabulary of classes in set theory, then one must look to the iterative conception itself. How can the conception account for the set-theoretical use of proper classes, when no such things as classes occur in the iterative hierarchy?

The actualist has an answer to this. Uzquiano's suggestion is to use Boolos' thesis about plural quantification being interdefinable with monadic second order logic, and that plural quantification does not commit one to the ontology of classes, to give an interpretation of the vocabulary of classes.

The idea from Uzquiano is to construe the reference to classes not to set-like collections that are not sets, but rather to pluralities of sets.[9] This means treating 'class' as a disguised plural reference.

Uzquiano points to Helen Cartwright for support for his suggestion. In "On Plural Reference and Elementary Set Theory", Cartwright leans on Russell's distinction from *The Principles of Mathematics* between 'class as one' and 'class as many' when she observes that in some uses, a term such as 'collection' "serves *only* to singularize a plural nominal."[10]

A relevant passage from Russell, is this:

> Is a class which has many terms to be regarded as itself one or many? Taking the class as equivalent simply to the numerical conjunction "A and B and C etc.," it seems plain that it is many; yet it is quite necessary that we should be able to count classes as one each, and we do habitually speak of *a* class. Thus classes would seem to be one in one sense and many in another.
>
> Bertrand Russell, *The principles of Mathematics*, (London: George Allen & Unwin, 1937, 2nd edition), 76

In *Principles* Russell concludes with drawing an ultimate distinction between 'class as one' and 'class as many', where the term 'collection' is defined as a class as many:

> By a collection I mean what is conveyed by "A and B" or "A and B and C," or any other enumeration of definite terms[11]

[9]ibid

[10]Helen Cartwright, "On Plural Reference and Elementary Set Theory" in *Synthese*, (1993), 213

[11]*Principles, 68*

And he points out a grammatical difficulty with the distinction made:

> A collection, grammatically, is singular, whereas A and B, A and B and C, etc. are essentially plural. This grammatical difficulty arises from the logical fact [...] that whatever is many in general forms a whole which is one; it is, therefore, not removable by a better choice of technical terms[12]

It is this "grammatical difficulty" Cartwright points to when she speaks of a term such as a 'collection', that "in its purely singularizing use, it affords means of referring in the singular to what can also be referred to in the plural".[13]

What Uzquiano proposes is to use the term 'class' as such a singularizing device. If this is possible, he says, then the truth of a sentence such as

(s) There are some sets that are such that no one of them is a member of itself and such that every set that is not a member of itself is one of them

will suffice for the truth of the sentence

(c) There is a class of sets such that no one of them is a member of itself and such that every set that is not a member of itself is one of the class.

Thus, for the actualist, the class-vocabulary can remain intact, while still no classes occurs in the set hierarchy. In this way, the actualist account for the class-talk in set theory.

Now, the situation is different for the potentialist, who claims there exists no actual plurality of *all* sets. Recall what Linnebo (2010) says about the implicit modal character of the quantifiers used in set theory.

> ...when a set theorist says that a formula holds for "all sets," she should typically be understood as claiming that no matter how far the hierarchy of sets is continued, the formula will hold for all the sets formed by then.[14]

The potentialist quite easily manage to talk about "all sets". However, what is referred to by 'all sets' is not all the sets there are. It is just all the sets that have been formed at the stage where you're at, which does not correspond to the use of V, M and j in set theory.

Recall that the principle of *Naive Plurality Comprehension* is rejected by the potentialist. The principle says that for any property P, there are the things that

[12]ibid, 70

[13]Cartwright, "On Plural Reference and Elementary Set Theory", 213

[14]Linnebo "Pluralities and Sets, 155

are P. In situations when the quantifiers in set theory is seen to have an implicit modal character, plural comprehension fails. Thus, given that the property P is that of being non-self-membered, the potentialist holds that there is not a determinate matter, what sets are non-self-membered.

For the potentialist to be able to explain how the iterative conception can account for the use of the proper classes in set theory, he must find a way to refer to *all* sets. But when it is an indeterminate matter what all sets are, it makes it a difficult case for reference.[15]

As a provisional conclusion, we can say that because the potentialist picture does not manage to give a satisfactory account for the use of class-talk in set-theory, it does not fulfill requirement 1. This means that its status as a motivation for the axioms of ZFC is weakened.

4.2 (2) An intuitive conception

Now, how are we to evaluate whether or not the two pictures attend to the intuitive character ascribed to the iterative conception? I think the best way of doing this is by simply evaluating to what extent the two different pictures actually are tenable interpretations of the iterative conception. In fact, one of the biggest worries connected to the actualist picture, is exactly whether or not its status as an aspect of the iterative conception is threatened.

The actualist's challenge:

We have seen that a consequence of making use of a generative vocabulary in explicating the iterative conception is that the conception upholds the *Principle of Naive Set Comprehension* (NSC), the thesis that whenever there are some things, the set of these things is formed. This means that it also sustains the principle of great importance to the potentialists, namely the principle holding that whenever there are some things, the set of these things *can* be formed. I will follow my use of terms from chapter 2 and call it *The Principle of Modal Set Comprehension* (MSC).

The potentialist holds MSC to be inherent in the iterative conception itself, and upholds the principle by a modal language. On the potentialist picture, there is no plurality of sets that cannot form a set. In this way, the potentialist picture seems to be a tenable interpretation of the iterative conception, since it gives a satisfactory account of the open-endedness of the iterative hierarchy.

[15]A suggestion from the potentialist is to rely on the full strength of second-order comprehension to be able to make sense of class-talk. However, this may account for some use of the vocabulary, but it is not probable that it can account for all such use.

However, a challenging objection to the actualist concerning requirement (2), is that the actualist, in his understanding of the generative vocabulary fails to uphold MSC, and thus must be seen as stopping the process of set formation. Linnebo, in Linnebo (2010), claims the most viable alternative to MSC, is to claim that some pluralities are too big to form a set. If he is right, and we assume that the actualist most probably would choose the most viable alternative available, the actualist is forced to accept a kind of limitation-of-size thesis and thus to discharge from the iterative path. Since, as we have seen, the iterative conception does not give any size restrictions as to what sets there are.

James Studd also claims that to explain why a plurality of sets do not form a set is a great challenge to the actualist, and that a proper explanation cannot be given within the framework of the iterative conception.

> ...I can see no hope for elaborating a nonmodal, tenseless stage theory – or for that matter, any other view – in order to meet this challenge, in a non-arbitrary and principled way.[16]

So, the claim from the potentialist is that, without any explanation as to why a rejection of the principle is plausible, the actualist picture cannot claim to be an aspect of the iterative conception of set.

This is a very serious charge against the actualist, but even so, the philosophical literature has not seen many answers or explanations as to why some pluralities do not form a set. I think the main reason for this is because, either, the actualist doesn't find it necessary to give an answer to this question, or he simply has no explanation to give. To see why this may be the case, it is useful to again look back to the actualist's understanding of the generative language used in explicating the iterative conception. The language of time and activity obviously gives the impression that the hierarchy of sets is open-ended. However, following Boolos, when the language of time and activity is treated as mere metaphors, the iterative conception can be explicated by a listing of the elements in the hierarchy (There are the null set,..., the set of those sets, the set of Those sets, etc.).

It becomes evident that on the actualist picture, the elements of the set hierarchy are all *there*, they have *already* been formed. And this is what an actualized hierarchy is like. Given a plurality of sets, the set of them is not an immediate possibility that *can* be actualized, it *is* actualized, and so are all the possibilities of set formation that occurs in the hierarchy. This view leads to paradoxes such as Russell's, if one does not put restrictions on what things form a set. Thus,

[16] James Studd, "The Iterative Conception Of Set, A (Bi-) Modal Axiomatisation", in *Journal of Philosophical Logic*, Vol. 42, (2013), 700)

the actualist picture holds that there are pluralities that don't form a set, such as the plurality of all sets.

It is important also to have in mind what was mentioned in chapter 3, that there is a difference between the actualist and the potentialist picture in how they view the relation between that of being a plurality and that of being a set. The actualist, contrary to the potentialist, holds there to be a substantial ontological gap between the existence of a plurality and a set. Thus, on this view, a plurality does not necessarily imply the existence of the set of that plurality. Again, as expressed by Boolos, "it doesn't follow just from the fact that there are some Cheerios in the bowl that, as some who theorize about the semantics of plurals would have it, there is also a set of them all."[17]

However, to this it is legitimate to ask what, in addition to the Cheerios in the bowl, is required for there to also exist a set of Cheerios? Boolos says nothing about this. One answer that has been given, is "Russell's paradox".[18] That means, there are certain facts about the world that are logical truths. That there is no set containing all sets that do not contain themselves is one of them. However, I agree with James Studd, who points to the fact that even though Russell's paradox shows there to be such logical facts, and the derivation of the paradox shows naive set theory to be inconsistent, the fact that there are such logical truths does not explain why certain sets, in this case, those that do not contain themselves, are not able to form a set.[19]

One may regard it an explanation, the fact that the actualist holds the elements of the plurality of all sets to be exactly that, namely *all* the sets there are. Thus, one may simply say that on the actualist account, some pluralities do not form a set, because there is no more sets to be formed. This answer, though, is just a reformulation of the problem in question: Why are there more pluralities in the world, than there are sets? If the elements of the plurality in question is coexistent, there is according to MSC, no reason as to why they should not form a set.

Assuming MSC to be a principle expressed by the iterative conception then, it looks as though the actualist has no explanation as to why some pluralities do not form a set. Thus, he is forced to acknowledge that on his own account, the process of set formation stops some arguably arbitrary place in the hierarchy.

However, this is not the final conclusion, since obviously, the actualist doesn't find the principle of MSC inherent in the structure of the iterative hierarchy. This is clear from Boolos' rejection of the generative vocabulary. The metaphors are just there to make vivid the distinctions between different posi-

[17]Boolos, "To Be is to Be a Value of a Variable", (1984), 72

[18]Such an answer is found in Richard cartwright, "Speaking of everything", in NoÃ»s (1994)

[19]Studd, "The Iterative Conception Of Set, A (Bi-) Modal Axiomatisation", 700

tions in the structure described by the iterative conception. When one looks at the explication of the iterative conception as just a listing of the elements it consists of, there seems to be nothing in this description that requires the relevant pluralities, such as the plurality of all sets, to form a set.

Thus, the potentialist objection to the actualist, that if the actualist cannot explain why some pluralities do not form a set, he contradicts the central idea of the iterative conception, is based on a presumption to which the actualist himself disagrees. The actualist can claim that, on the iterative conception, it is not the fact that all pluralities form a set, and thus, to claim that there are such pluralities, and provide no explanation as to why these pluralities behave like they do, does not contradict any idea expressed by the iterative conception.

So, the essential disagreement between the actualist and the potentialist is captured in the question of whether or not MSC is expressed by the iterative conception. The potentialist holds MSC to be read off from the conception, the actualist disagrees. The important question is thus whether or not the actualist's rejection of MSC implies that the actualist picture fails to be a picture of the iterative hierarchy.

I claim that it doesn't. I believe the actualist's rejection of NSC *and* MSC is unproblematic on the assumption that there is a substantial ontological gap between the existence of a plurality and the existence of a set. Thus, if one rejects the actualist picture as a satisfactory interpretation of the iterative conception, one would have to give a reasonable account of why this assumption is wrong. It is possible that this can be done, but will in any case not be done here.

However, one may still wonder why, on the actualist picture, the relevant pluralities do not form a set. The actualist gives no explicit explanation as to why it is the case, and one is left with the impression that it is viewed as a primitive fact of the matter, that is beyond explanation. As will be discussed in connection to requirement 3, this is a problematic feature of the actualist picture, but it doesn't prevent the actualist picture to give a plausible interpretation of the iterative conception.

Thus, the conclusion to draw regarding requirement 2 is that both pictures are tenable interpretations of the iterative conception. In this way, they both provide an independent feature to the motivational aspect of the iterative conception, and account the "thought behind" the axioms, that the conception is claimed to be.

4.3 (3) Explaining paradox

It was argued in chapter 1 that the iterative conception provides an answer to the set theoretic paradoxes because it *explains* why there are no paradoxical sets, instead of just disallowing the sets to occur.

It is in connection to this last requirement that I find the actualist picture to meet a serious challenge. We have seen that the actualist does not hold the *principle of modal comprehension* (MSC) to be expressed by the iterative conception, for the reason that, on the actualist account, there are some pluralities that do not form a set.

As we saw, the actualist gives no explanation as to why this is the case, but just holds that it *is* the case. On the assumption that pluralities are something substantially ontologically different from sets, it seems intelligible to hold that some things are just these things, and nothing more. Must the actualist really provide a reason for why this is the case? There are several examples from the physical science, showing that things are said to be in a specific way, even though the standard model of particle physics, gives no explanation as to why this is the case. For instance will physicists claim that there exist three generations of particles which are identical, but have different mass. The standard model gives no explanation as to why there are three such generations.[20]

On the other hand, are primitive facts in mathematics even comparable with such primitive facts in physics? First of all, facts like the one above, about the unexplainable three generations of particles, are still facts that physics hope to be able to explain some time in the future. This cannot be said about the pluralities that do not form a set. Also, it is perhaps reasonable to require something more from a mathematical explanation than from an explanation in the physical science. Mathematics does after all play a central role in the physical scientific characterization of the world.

Thus, that the actualist does not provide an explanation as to why some pluralities form a set may be problematic itself. However, I'm not able to discuss how problematic it may be here. The relevance of the lack of such an explanation here is related to requirement 3, since it is quite obvious that this lack of explanation affects the actualist's response to the set-theoretic paradoxes.

To see how, we can imagine a philosophy student S asking the actualist A out on the matter in question:

S: Why do sets like the Russell set and the universal set not occur on the iterative conception?

A: Well, that's because the structure of the iterative conception does not allow them to occur.

S: Ok, and why is that?

A: On this way of thinking about sets, every set is formed at some earliest stage, and has as members only the sets (or individuals) formed

[20]Everything we see around us consist mainly of first-generation particles, and the higher generations are only perceptible on higher levels of energy

at earlier stages. This means that, for a set to be formed, its elements has to be "available" for set formation. The reason why the paradoxical sets do not occur on this description is because the elements are never available for set formation.

S: But what makes them unavailable?

A: What makes them unavailable is the fact that the set of them doesn't exist on the iterative conception. Since no set can be available on an earlier stage than on the stage where it is formed itself, no set contains itself.

S: Ah, I see, for the elements of a set to be available, they would have to be "all there", which is not the case with the elements *all sets*, since they are missing a set, or... hm, that doesn't seem right either, they are after all supposed to be *all* sets there are. What am I missing here?

A: I think what puzzles you is the fact that the iterative conception holds *all sets* to be existent, but claims the set of them not to occur. Well, whenever you have some things, does the fact that these things exist, implies that the set of them also exist? For instance, when you're having your cheerios in the morning, do you think you're eating a set of cheerios?

S: No, not really.

A: Exactly!

What we see is that the question as to why the paradoxical sets do not occur on the iterative conception, does at some point necessarily lead to the question of why the relevant pluralities do not form a set. Since the actualist has no explanation as to why this is the case, and must claim that it is a primitive matter of fact, he is bound to hold that the reason why the paradoxical sets do not occur on the iterative conception ultimately just is a matter of fact – it is beyond explanation. And this does not satisfy requirement 3.

Now, the potentialist does, as the actualist, holds the structure described by the iterative conception to be responsible for the fact that the paradoxical sets do not occur. However, the last part of the conversation above would look somewhat different if S were asking the potentialist P out on the matter:

S: Why do sets like the Russell set and the universal set not occur on the iterative conception?

...

S: Ah, I see, for the elements of a set to be available, they would have to be "all there", which is not the case with the elements *all sets*, since they are missing a set, or... hm, that doesn't seem right either, they are after all supposed to be *all* sets there are. What am I missing here?

P: You're not really missing anything. It is the case that *all sets* cannot be "all there" together, and that this fact accounts for this plurality's unavailability to form a set. What makes it all puzzling is that in your reasoning you're assuming that it is a determinate matter what are *all sets*. However, this is not the case on the iterative conception, since the structure it describes is a potential structure.

S: Alright, so because you never have all the sets there are, there is no question as to why they do not form a set. Well, that makes sense.

Thus, because the potentialist gives an explanation as to why there exist no pluralities that are unable to form a set on the iterative conception, he provides a satisfactory response to the set-theoretical paradoxes and satisfy requirement 3.

4.4 Two tenable interpretations

Now, from what has been said above, it is obvious that both the actualist and the potentialist picture faces a difficult challenge, and thus that the iterative conception looses one or another of its alleged benefits on each interpretation. On the potentialist account, the conception's role as a motivation for the axioms of ZFC is weakened, even though it still can be said to pinpoint a "thought behind" them. While, on the actualist account, the iterative conception fails to give a satisfactory response to the set-theoretical paradoxes. Which one of these sacrifices is worst to make for the iterative conception remains to be investigated, but some remarks will be made below.

However, what must be discussed here, is what we have seen above, that two radically different pictures of the set hierarchy can both be tenable interpretations of the iterative conception. To understand this, it is useful to look back at the historical development of the understanding of the infinite from the previous chapter. We saw that the meaning of the traditional concept of infinity from the Ancients changed with the Cantorian revolution, from that of being *withouht a limit*, which means to be *unbounded*, to that of being what is *unbounded by any natural number*. Unboundedness for the Ancients, however, was equivalent with that of being unbounded by any natural number.

The time after Cantor saw new and bigger numbers; that meant new and bigger *measurements*. These new measurements – the transfinite numbers – actualized the realm of the transfinite. As was also discussed in chapter 3, there is a tension in Cantor's account of the Absolute, about whether one is to understand also the Absolute as a definite domain, or rather treat it as an inconsistent multiplicity. If one choose to interpret Cantor such that the Absolute is an "all-encompassing universe"[21], the Cantorian revolution must be seen as an actualization also of the Absolute.

The actualist's understanding of the iterative hierarchy can be seen as a development of this actualized way of interpreting Cantor's different infinite realms. What distinguishes Cantor's Absolute from the transfinite on the actualist reading is the fact that the Absolute is beyond both increasability and number. What this difference amounts to however, is not obviously clear when considering the fact that the Absolute also is considered a definite domain. Similarly, The actualist holds the iterative hierarchy to be a plurality, both unmeasurable and unincreasable, but still something definite. Thus, as on the actualized interpretation of Cantor, the hierarchy is on the actualist account treated very similar to the other infinite sets.

Now, it is interesting to compare this developmental understanding of Cantor's Absolute, with the potentialist's explicit use of the notion of an inconsistent multiplicity. Because, also on the potentialist account, the division between the sets in the hierarchy and the hierarchy itself marks the division between what is measurable and unmeasurable. But here, the division resembles the ancient distinction between the finite and the infinite, and seems much more fundamental than on the actualist account.

Linnebo notes in Linnebo (2013b), that the ideal development of the concept of the infinite after the Cantorian revolution would have been to regard something as infinite "just in case it cannot be bounded by *any* measuring stick".[22] This suggests a place-shift for the infinite in the iterative hierarchy.

> Since "finite" originally meant "bounded" or "measurable", the result of discovering a new system of longer measuring sticks should be to regard more things as finite – albeit in a generalised sense. On this alternative conceptual development, Cantor's new numbers would have been regarded not as infinite but as a generalisation of the finite.[23]

This suggestive shift of the infinite's place in the hierarchy, reflects the potentialist interpretation of the iterative conception. The infinite sets, together with the finite, constitute the actualized part of the set-hierarchy – the measurable part. The hierarchy itself, however, is intimately connected to Aristotle's

[21]Jané, "Idealist and realist elements in Cantor's approach to set theory", 22
[22]Linnebo, "What is the infinite", (2013), 46
[23]Ibid

potential infinite, which we recall has this mode of existence: "one thing is always being taken after another, and each thing that is taken is always finite, but always different"[24].

Thus, we see that the main difference on the two understandings of the infinite, is about what it means to be *unmeasurable*. The actualist holds that the actual plurality of all sets can manage to represent the unmeasurable hierarchy of sets, while the potentialist claims that it is not good enough. And here I agree with the potentialist. The unmeasurability of the hierarchy must be taken at face value, which is not done by the actualist. Since, how can something with no end to it be represented by an extensional plurality?

To the actualist defense, one can object by saying that the open-endedness of all the other infinite sets in the hierarchy are bounded. With this in mind, it cannot be controversial to suppose the hierarchy itself to be an actualized plurality. To this I have no good answer. I do however, think the hierarchy's property of being both beyond number and increasability should mark a distinction between the hierarchy and the other infinite sets, but on the actualist account, no such clear distinction is made.

[24] Aristotle, *Physics*, book III, ch.6, 348

Conclusion

The motivation behind this investigation was to gain an understanding of what the actualist and potentialist interpretations of the iterative conception is like; what their main differences are and how these affects the iterative conception's status as a beneficial conception of set.

We have seen that three requirements where put forth, for holding the iterative conception to function as a motivation and "thought behind" the axioms of ZFC, and to provide a satisfactory response to the set-theoretical paradoxes. It corresponds to an actual conception, it is intuitive and it provides an *explanatory* response to the set-theoretical paradoxes.

The problematic aspect of the potentialist picture is how it can make sense of proper class talk in set theory. Because the hierarchy on the potentialist account is characterized as existing potentially, and where the existence of a set is an immediate possibility given its elements, it is not possible to talk about *all sets* and actually refer to all the sets there are. The actualist picture, on the other side, faces a problem when it is to provide an explanatory response to the paradoxes. The picture gives us a description of what sets occur and what sets do not occur. However, since the generative vocabulary of the iterative conception is to be understood merely as metaphors, what sets are *available* for set formation, can only be understood as the pluralities that *are in fact formed* to a set. However, this gives us no reason as to *why* the relevant pluralities cannot form a set, it is only a statement holding that they're not.

Then, letting the benefits ascribed to the iterative conception function as a standard, which picture provides the most preferable aspect of the conception? As we have seen, the iterative conception, on each interpretation, sacrifices something arguably valuable. To correspond to an actual set-theoretic practice where claimed to be of importance for the iterative conception by both Gödel, Martin and Boolos. However, to what extent must the conception correspond? Is the conception an absolutely unsatisfactory motivation for the axioms of ZFC if it cannot make full sense of class-talk in set theory? I believe some philosophers will think that it is, but not only for the reason that the conception weakens its position as a motivation for the axioms of ZFC. Some philosophers celebrate science to that extent that the lack of correspondence such as in the potentialist case, will count as a reason for rejecting the theory. At least when there is another conception available, such as the actualist one, that perfectly well can make sense of the vocabulary of classes in set theory.

I think this is a hasty conclusion. It may as well be looked upon as courageous of the potentialist, to exercise some philosophical autonomy. However, our focus here is on the iterative conception as a motivation for the axioms of ZFC. It seems obvious that the potentialist picture is weakened as a motivation for the axioms of ZFC, especially considering that the actualist picture is available. But on the actualist account, the iterative conception fails to give a satisfactory response to the set-theoretic paradoxes, which seems to be an essential quality of the iterative conception.

Surely, I do think it's possible to strongly prefer the one picture over the other on the reasons of these different deficiencies, and I think, after thorough investigation of the matter, one will be able to take a qualified decision. Unfortunately, this is more than I can do here. Thus, I will leave it an open, but further question, whether the iterative conception does better as a weaker motivation for the axioms of ZFC, but with the explanatory power of giving a satisfactory response to the paradoxes, or as a strong motivation for the axioms, but without such an explanatory power.

Despite their deficiencies, both the actualist and potentialist picture must be seen as tenable interpretations of the iterative conception. And most importantly, the discussion in the previous chapter revealed that their apparently radically different characterizations of the iterative hierarchy, stems from their disagreement about how one are to understand the unmeasurable character of the infinite. The actualist picture holds the unmeasurable character of the hierarchy to be understood as the infinitude of the other infinite sets is understood, but only with the exception that on the hierarchy, there is no measuring sticks available. The potentialist disagrees, and holds the unmeasurability of the hierarchy not just to be the lack of bigger measuring sticks, but to be of a completely different character than the other infinite sets.

I believe the potentialist picture, with its characterization of the iterative hierarchy, captures the intuitive understanding of what it means for something to be unmeasurable. For this reason, I think the potentialist picture is the most plausible interpretation of the iterative conception. I do also, however, believe that my investigation has showed that the two pictures is not so radically different interpretations of the conception that was first expected. As we have seen, the ontological difference between the potentialist and actualist hierarchy is simply the consequence of two different readings of the open-ended character of the hierarchy.

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